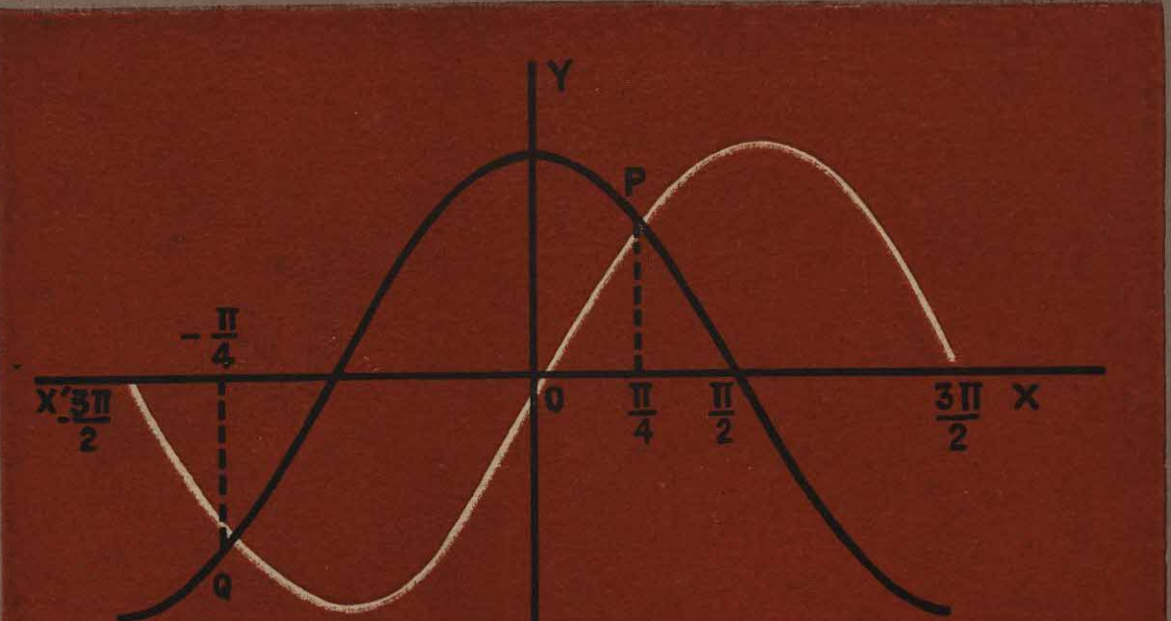
HIGHER SECONDARY MATHEMATICS -2

D.K.SINHA & A.B.ROY



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HIGHER SECONDARY MATHEMATICS

Volume 2

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Preface

THE immediate aim of *Higher Secondary Mathematics*, Volume II, the companion to Volume 1, is to serve as the basic textbook for students of Class XI who have chosen Mathematics as an Elective subject in the General Stream Courses of study formulated by the West Bengal Council of Higher Secondary Education.

This volume has been written according to Paper II of the Mathematics syllabus. The prescribed topics in Elements of Differential Calculus, Integral Calculus, Differential Equations, and Mechanics—Statics and Dynamics—have been thoroughly and intensively dealt with.

Higher Secondary students have ten years of schooling behind them. In some of these schools, students are made familiar with what is termed 'traditional' mathematics, while in some 'modern' mathematics is taught. As for Volume 1, here too logic and reasoning are the cornerstones of our approach and thus pupils from either backgrounds will have no difficulty in using this book.

The numerous examples follow directly from the principles stated. These carefully thought-out examples, based on real life situations, especially within our students' realm of experience, relate theoretical formulations with practical problems. We have continued to use the method followed in the earlier volume so that in this book too both the teacher and the more interested student will be able to develop further on the examples given here—as objective and other types of exercises can be built around these examples. These examples will therefore develop competition skills, enabling students to face tests in mathematical ability with confidence and equanimity. The cooperation and sincere hard work of our team of authors have made this venture possible. We hope teachers and pupils will find this volume to be as useful and interesting as we have tried to make it. All comments and suggestions will be warmly appreciated.

D. K. SINHA
A. B. ROY

Contents

Preface iii

PART ONE : **Differential Calculus**

1. FUNCTIONS AND GRAPHS 3
2. LIMIT AND CONTINUITY 13
3. DERIVATIVES 25

PART TWO : **Integral Calculus**

4. INTEGRATION OF FUNCTIONS 53
5. METHODS OF INTEGRATION 57
6. DEFINITE INTEGRALS 63
7. APPLICATION OF INTEGRAL CALCULUS (QUADRATURE) 68

PART THREE : **Differential Equations**

8. FIRST ORDER DIFFERENTIAL EQUATIONS 73
9. SECOND ORDER DIFFERENTIAL EQUATIONS 78

PART FOUR : **Mechanics**

10. PRELIMINARY IDEAS 85

PART FIVE : **Dynamics**

11. KINEMATICS : VELOCITY 89
12. ACCELERATION 103
13. NEWTON'S LAWS OF MOTION 117
14. SIMPLE HARMONIC MOTION 124
15. TWO-DIMENSIONAL MOTION 131

PART SIX : **Statics**

16. CONCURRENT FORCES 143
17. PARALLEL FORCES 152
18. MOMENT OF A FORCE 156
19. COUPLES 161
20. CENTRE OF GRAVITY 166
21. NON-CONCURRENT COPLANAR FORCES 178
22. GRAPHICAL STATICS 182
23. FRICTION 189
24. MACHINES 197
Answers to Exercises 221

Erratum

Page	*Line*	*Read*	*For*
56	6	$\int\left\{\frac{\alpha^{3x}+\alpha^{2x}+\alpha^{x}}{\alpha^{4x}}\right\}dx$	(ix) $\int\left\{\frac{\alpha_3{}^{x}+\alpha_2{}^{x}+\alpha^{x}}{\alpha_4{}^{x}}\right\}dx$

DIFFERENTIAL CALCULUS

1 Functions and Graphs

CALCULUS, historically speaking, grew out of problems concerning speed, such as what speed is and how we can measure it. One can thus trace its origin to the early days of the seventeenth century and from then, almost every development in science and mathematics upto the twentieth century had something to do with calculus. The great mathematicians of the seventeenth century, Newton and Leibnitz sought to put the study of calculus in a mathematical setting and what we shall be studying in the following chapters are largely their contributions. In recent times, calculus, as it has happened with other branches, has assumed rigorous forms that often go by the names like Mathematical Analysis. We would like to introduce the basic concepts of calculus on the basis of an idea about some number systems and allied topics. We shall do this first in some of the following sections before proceeding to take up problems of calculus.

1.2 Some preliminary ideas

1.2 (*a*) *Rational Numbers*

The concept of number owes its origin to the counting of objects. Counting resulted in the numbers 1, 2, 3 and so on, which are now termed **natural numbers.** Later came the concept of what are called **rational numbers** like $\frac{1}{2}$, $\frac{3}{4}$, etc. These were upshots of measurements of continuous quantities (lengths, weights, etc.). The negative integers (−1, −2...) and zero came into mathematics with the development of Algebra and we could then think of **negative rational numbers,** $-\frac{3}{2}, -\frac{1}{2}\ldots$, etc. All these numbers can be put in a simple form p/q where p and q are integers. We can take q to be a positive integer and the rational number p/q is positive or negative according as p is a positive or negative integer. If $q=1$, the rational numbers become an integer. For example, $\frac{4}{1}, \frac{5}{1}, \frac{6}{1}$, etc. are rational numbers.

1.2 (*b*) *Real Numbers*

In practice, measurements are carried out by means of instruments. The result of a measurement is often roughly expressed as some rational number. But the fact remains that rational numbers are not quite adequate to give exact measurements. That is to say, no rational number is capable of precisely expressing the length of, say, the diagonal of a square, if its side is taken as the unit of measure. Neither can a rational number express exactly the sine of 60° angle, the cosine of a 22° angle, the tangent of a 17° angle, the ratio of the circumference to the diameter of a circle, etc. Speaking generally, it is not possible to express the ratio of incommensurable segments exactly by means of a rational number.

Thus if we seek to express in some exact terms the ratio of incommensurable segments, we need to introduce new numbers that are called **irrational numbers.** An irrational number expresses the length of a segment that is not commensurable with the unit of numbers (scale unit). The rational and irrational numbers taken together form the totality of **real numbers.** Real numbers suffice to express with exactitude the length of any segment.

An irrational number cannot be exactly equal to any rational number, but it may be possible to find for every irrational number, a rational number (say a decimal) which is approximately equal to the irrational number (too large or too small) and which can be made to approach it to any arbitrary degree of accuracy.

How can we visualise this geometrically? To do this, we have to introduce what is called a number line.

1.2 (*c*) *The Number Line*

On the (straight) line (Fig. 1.1), choose an origin O, a scale unit OA and a positive direction (say from X' to X). Then every real number x will correspond to a definite point M, the distance of which from O is equal to x.

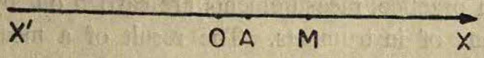

Fig. 1.1.

Numbers are represented in this way by points on a line. The line $X'X$ on which the points are specified is called the **number line** or often the **number scale,** or **number axis.**

1.2 (*d*) *Variable and Constant Quantities*

A **variable (quantity)** is a quantity which can take on different values within the framework of a given problem. In contrast, a **constant (quantity)** is one which, within the framework of a given problem, has one and only one value. One and the same quantity can be a constant in one problem and a variable in another.

In most physical problems, the boiling point T of water is a constant quantity. But when one has to take into account changing atmospheric pressure, T becomes a variable quantity. In the equation of the parabola $y^2 = 4ax$, the coordinates x, y are variables. The parameter a is a constant if we consider only one parabola. But if we consider a set of parabolas with a common x-axis and a common vertex O, then the parameter a is a variable quantity.

Variables are ordinarily denoted by the last letters of the alphabet (x, y, z, u, v, w......) while constants by the first letters (a, b, c......).

1.2 (*e*) *Absolute value of real numbers*

The absolute value of a real number a, denoted by $|a|$ is defined as $+a$ if $a > 0$, $-a$ if $a < 0$ and 0 if $a = 0$. For example,

$$|-5| = -(-5) = 5, \quad |+2| = 2,$$
$$|-\tfrac{3}{4}| = \tfrac{3}{4}, \quad |0| = 0$$

Some simple and easy properties of absolute values may be given as

(i) $|ab| = |a|\,|b|$, $|abc| = |a|\,|b|\,|c|$

(ii) $|a+b| \leqslant |a| + |b|$, $|a+b+c| \leqslant |a| + |b| + |c|$

(iii) $|a-b| \geqslant |a| - |b|$

(iv) $|a-b| = |b-a|$

where a, b, c are real numbers.

1.2 (*f*) *Sets*

A **set** is a collection of objects having well defined characteristics. If we say R is a set of real numbers, we mean R is a collection of all rational and irrational numbers and is denoted, symbolically, as $R = \{x\}$ where x assumes any value of the real number system. Similarly, the set of integers can be denoted as

$$I = \{\pm 1, \pm 2 \ldots\} = \{x\}$$

where x assumes any value of positive and negative integers. The individual objects of a set are called **members** or **elements.** Hence x in the set I is a member of that set or in other words $+1$, $+2$, -1, -2......are members of the set I. A **singleton set** consists of one element only. A set having no element is called an **empty** or **null** or **void set.**

In the following discussions we shall consider the variable x as a real number or elements of a set as real numbers unless otherwise stated.

1.3 Function

If for every value of a variable x, which belongs to some collection (set) E, there corresponds one and only one finite value of the quantity y, then y is said to be a function (single valued) of x or a dependent variable defined on the set E. x is the argument or independent variable. The fact that y is a function of x is expressed, in brief form, by the notation $y=f(x)$ or $y=F(x)$ and the like.

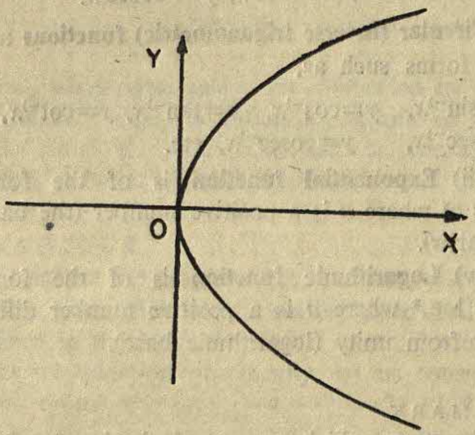

Fig. 1.2

Geometrically, we can represent a function by a curve obtained by the set of points (x, y) in the xy-plane (Fig. 1.2).

REMARK

If to every value of x belonging to some set E there corresponds several values of the variable y, then y is called a multiple valued function of x defined on E. From now on we shall use the word function only in the meaning of a single valued function, if not otherwise stated.

For example, the area A of a square is a function of its side x and is given by $A=x^2$. A is a single valued function or simply a function of x because for every value of x we get one and only one value of y. This means that geometrically corresponding to a set of values of (x, y) there corresponds a point on the curve representing the function. Again, if $y^2=x$ where $x>0$ then to each x there correspond two values of y. Hence y is a double valued function of x. We can consider this as two single valued functions f and g where $f(x)=\sqrt{x}$ and $g(x)=-\sqrt{x}$. These obviously correspond to two parts of the curve, representing the function one above the x-axis and the other below it.

We say that $f(x)$ is the value of the function f for some value of x. So, for a given function $f(x)=x^2$, we have its value for $x=2$ is $f(2)=2^2=4$. This corresponds to the point $(2, 4)$ on the curve representing the function.

If for any value of x, the value of function representing $f(x)$ is meaningful then the function is defined for that value of x or at the point x. Accordingly a function is defined for all values of x if $f(x)$ is meaningful for all values of x. For example, $y(=f(x))=x$ is a function defined for all real values of x, but $y(=f(x))=\dfrac{x^2-16}{x-4}$ is not defined at $x=4$ because for $x=4$, $f(x)=\frac{0}{0}$ which is meaningless.

1.3 (*a*) *Domain of definition of a function*

The collection of all values which (under the conditions of the problem at hand) the variable x of a function $f(x)$ can assume is called the **domain of definition** or simply, **domain of the function.**

In the simplest cases, the domain of a function is either a **closed interval** denoted by $[a,b]$, which is the set of real numbers x satisfying the inequalities $a \leqslant x \leqslant b$ or an **open interval** denoted by (a, b) which is the set of real numbers satisfying the inequalities $a < x < b$.

For example, the function y given by

$$y=\sqrt{x-2}+\sqrt{7-x}$$

is meaningful only for $2 \leqslant x \leqslant 7$. Hence the domain of definition of the given function is the set of all numbers from 2 to 7 including the boundary points.

As another example, let us look for the domain of definition of the function

$$y=\sqrt{\sin 2x}$$

The function is defined if $\sin 2x \geqslant 0$, i.e. if $2m\pi \leqslant 2x \leqslant 2m\pi + \pi$, i.e. if $m\pi \leqslant x \leqslant m\pi + \frac{\pi}{2}$, $m = 0, \pm 1, \pm 2 \dots$

Hence the domain of definition of the given function is the set of values of x satisfying the inequality $m\pi \leqslant x \leqslant m\pi + \pi/2$.

The function

$$y=\frac{1}{\sqrt{x^2-1}}$$

is defined if $x^2-1>0$, i.e. if $x>1$ and $x<-1$. Thus the domain of the given function is the set of two intervals :

$$-\infty < x < -1 \quad \text{and} \quad 1 < x < +\infty$$

1.3 (*b*) *Classification of functions*

(i) As already mentioned, functions may be either **single-valued** or **multiple-valued.** For example, $y=x$ and $y^2=x$ are respective examples of single-valued and multiple-valued functions.

(ii) Functions are often said to have explicit and implicit forms. For an **explicit function**, we express it as $y = f(x)$ while **implicit functions** are represented as $f(x, y) = 0$.

(iii) Functions may be classified as elementary and non-elementary. An **elementary function** is a function which involves a limited number of elementary operations. For example, the functions

$$y=\frac{3x^2}{1+\log x}, \; y=\log \sin^3\sqrt{1-3\sin x},$$
$$y=\log\log(3+2\sqrt[3]{\sin x})$$

are elementary. Functions which cannot be expressed in this manner are termed **non-elementary.** .

1.3 (*c*) *Basic Elementary functions*

We often distinguish between different forms of elementary functions :

(i) **Power functions** are the form $y = x^n$ (where n is a constant real number). For $n = 0$, the power function is a constant quantity ($y = 1$). A combination of them in the form such as $y = ax^2 + bx + c$, $y = ax^4 + bx^3 + cx^2 + dx + e$, etc. are known as **algebraic functions.**

(ii) **Trigonometric functions** are the forms such as,

$$y = \sin x,\; y = \cos x,\; y = \tan x,$$
$$y = \cot x,\; y = \sec x,\; y = \operatorname{cosec} x.$$

Circular (inverse trigonometric) functions are the forms such as,

$y=\sin^{-1}x$, $y=\cos^{-1}x$, $y=\tan^{-1}x$, $y=\cot^{-1}x$, $y=\sec^{-1}x$, $y=\operatorname{cosec}^{-1}x$, etc.

(iii) **Exponential function** is of the form $y = a^x$ where a is a positive number (the base number).

(iv) **Logarithmic function** is of the form $y = \log_a^x$ where a is a positive number different from unity (logarithmic base).

REMARK

Functions which are not algebraic are often called **transcendental functions.**

1.3 (*d*) *Inverse Function*

If from $y = f(x)$ we can write $x = \phi(y)$, then the function $\phi(y)$ is called an inverse function of $f(x)$.

For example, the function $y=x^2$ can be written as $x=\pm\sqrt{y}$ which is the inverse of the function $y=x^2$ and is evidently double-valued; (what would be its geometrical representation ?)

The inverse of the function $y= \sin x$ is the multiple-valued function $x=\sin^{-1}y$ (defined for all values of $-1<y<1$). What will be its geometrical representation ?

1.3 (*e*) *Graph of a function*

The graph of a function is a pictorial representation of a function on rectangular coordinate system. A set of points (x,y) in an xy-plane, whose coordinates are connected by the equation $y=f(x)$, is called the graph of the given function (Fig. 1.3). Let us construct graphs of some elementary functions.

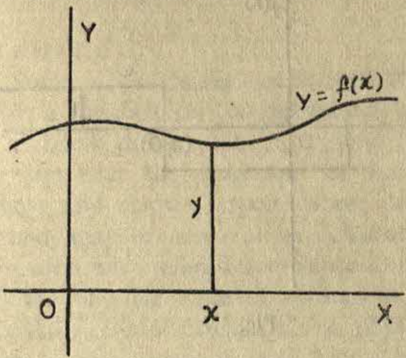

Fig. 1.3

EXAMPLE 1

Construct the graph of the function $y=x+3$.

The given function is single-valued and is defined in the interval $-\infty < x < \infty$. To construct the function graphically, let us consider some values of x say,...$-4, -3, -2, -1, 0, 1, 2, 3$...and determine the values of y from $y = x + 3$ for each value of x. This gives the set of points...$(-4, -1)$, $(-3, 0)$, $(-2, 1)$, $(-1, 2)$, $(0, 3)$, $(1, 4)$, $(2, 5)$, $(3, 6)$, etc. in the xy-plane. We now plot the points referred to a rectangular coordinate system xy (shown in Fig. 1.4) and then join the points with a line. This line must be extended to infinity on either direction as the function is defined for every value of x between $-\infty$ to ∞. Hence the graph of the given function is a straight line of the form $y=mx+c$ where $m=1$ and $c=3$ represented in xy frame.

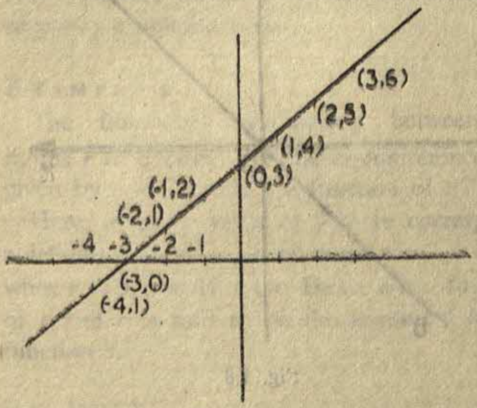

Fig. 1.4

EXAMPLE 2

Draw the graph of the function $f(x) = x^2$. The function is defined for all values of x in the interval $-\infty < x < +\infty$. A table giving values $f(x)$ for various values of x is given below.

TABLE 1-1

x	0	1	-1	2	-2	3	-3
$f(x)$	0	1	1	4	4	9	9

We now plot points $(0,0)$, $(1,1)$, $(-1,1)$, $(2,4)$, $(-2,4)$, $(3,9)$, $(-3,9)$ in $(x,f(x))$ frame of reference, and join the points by a line. This set of points defines a curve which is the part of a parabola (Fig. 1.5).

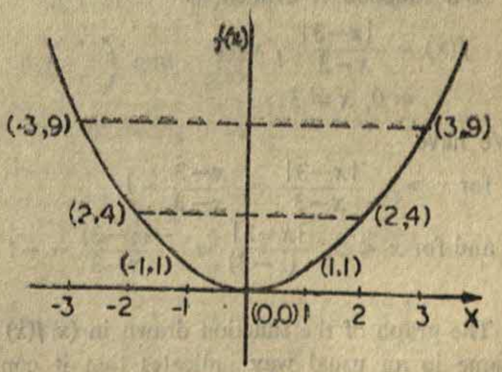

Fig. 1.5

1.3 (*f*) *Graph of arbitrarily defined functions*

A function may be defined arbitrarily. For example

$$f(x)=x^2 \quad \text{if } x\neq 2$$
$$=0 \quad \text{if } x=2$$

This shows that the function is defined for all real values of x. To draw the graph we consider some values of x in $f(x) = x^2$, $x \neq 2$. The set of points $(x, f(x))$ thus obtained when plotted in $(x, f(x))$ frame and then joined gives parts of the parabola $f(x)=x^2$ indicated in Fig. 1.6 by curves *AOB*, *CD* and the point (2,0). It appears from Fig. 1.6 that the graph contains what is called a **break** at $x=2$.

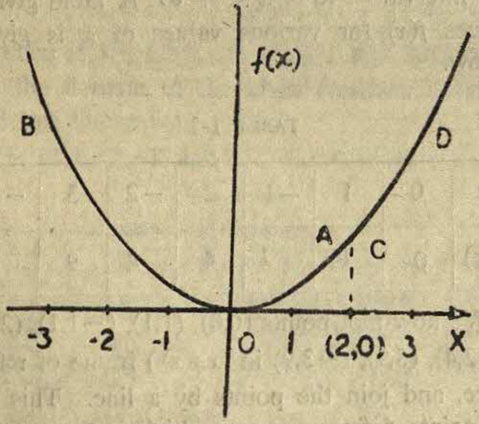

Fig. 1.6

If a function is defined as

$$f(x) = \frac{|x-3|}{x-3}, \; x\neq 3$$
$$= 0, \; x = 3$$

We have,

for $x > 3$, $\frac{|x-3|}{x-3} = \frac{x-3}{x-3} = 1$

and for $x < 3$, $\frac{|x-3|}{(x-3)} = \frac{-(x-3)}{x-3} = -1$

The graph of the function drawn in $(x, f(x))$ frame in an usual way indicates that it consists of line $y=1$, $x>3$; $y=-1$, $x<3$ and the point (3,0). This also shows that the graph has a break at $x = 3$ (Fig. 1.7).

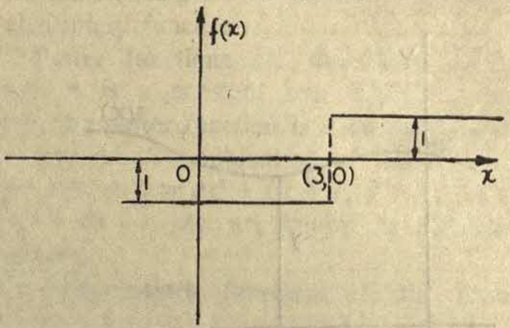

Fig. 1.7

EXAMPLE 1

Find graph of the function

$$f(x) = \frac{x^2}{x}, \quad x\neq 0$$
$$= 0, \quad x = 0$$

The function clearly indicates that $f(x)=x$ for $x\neq 0$ and $f(x)=0$ when $x=0$. The graph of the function drawn in an usual way as above is shown in Fig. 1.8. This figure is actually a straight line *AOB* with no break at any point.

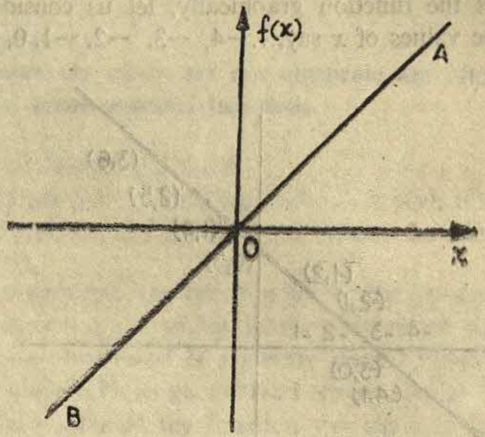

Fig. 1.8

It is to be noted that the prescribed value $f(x) = 0$ at $x = 0$ removes the undefinedness of the function $f(x)=\frac{x^2}{x}$ for all x. In other words, $f(x)=\frac{x^2}{x}$ has been made defined at $x = 0$ by prescribing its value $f(0) = 0$.

REMARK

Some of the examples discussed above show how a function can be represented arbitrarily. The break that appears in some of the graphs reveals that the behaviour of function $f(x)$ above and below the point where the break is found, may or may not be different. Such a situation has a close resemblance to the change of the physical state of substances. Loosely speaking, a sudden change in a physical situation is often shown by a kirk or a break with mathematical representation of the situation.

1.4 Illustrative examples

EXAMPLE 1

A ball is rolling on the ground, s is the distance from the point of hit, t is the time elapsed after the hit. How would you express s as function of t?

The quantity s is a function of t, i.e. $s = f(t)$ because the ball crosses a definite distance at every instant of rolling. Here t can also be expressed as a function of s, i.e. $t = f(s)$ for, to every distance traversed by the ball there corresponds a definite time.

EXAMPLE 2

The functional relationship between the radius r of a circle and the circumference s is given by $s = 2\pi r$. Is s a function of r?

Here, to every value of r there corresponds a definite value of s. Such when $r = 2$, $s = 4\pi$; when $r = 5$, $s = 10\,\pi$, etc. Hence s is a function of r and r is said to be the **argument of the function** s.

EXAMPLE 3

Find the domain of definition of the function $y=\frac{1}{x}$. The formula $y=\frac{1}{x}$ is meaningful for all values of x except when $x=0$. So the domain of definition of the given function is the set of all real numbers except 0.

EXAMPLE 4

Show that the function $s = 1 + 2 + 3 + \ldots\ldots + n$ is an elementary function.

The given function is an elementary function because it can be expressed as $s = \frac{n(n+1)}{2}$, which contains a limited number of elementary operations.

EXAMPLE 5

Which of the functions $y=x^2+5x+3$ and $x^3-xy^3+3y^2+2=0$ is explicit or implicit?

The function $y=x^2+5x+3$ is an explicit function because this function is of the form $y=f(x)$. The second function $x^3-xy^3+3y^2+2=0$ is an implicit function because it is difficult to express the second one in the form $y=f(x)$.

EXAMPLE 6

If $f(x) = x^2 + 6x - 4$, verify $f(1) = 3$,
$f(3) = 23$

Here $f(1) = 1^2 + 6.1 - 4 = 1 + 6 - 4 = 3$
$\therefore\ f(1) = 3$
Again $f(3)=3^2+6.3-4=9+18-4=23$
$\therefore\ f(3) = 23$

EXAMPLE 7

If $\phi(x) = \frac{x-1}{3x+5}$, write the expression $\phi\left(\frac{1}{x}\right)$ and $\frac{1}{\phi(x)}$.

Here replacing x by $\frac{1}{x}$ in $\phi(x)$ we have

$$\phi\left(\frac{1}{x}\right) = \frac{\frac{1}{x}-1}{\frac{3}{x}+5} = \frac{1-x}{3+5x}$$

Again, $$\frac{1}{\phi(x)} = \frac{1}{\frac{x-1}{3x+5}} = \frac{3x+5}{x-1}$$

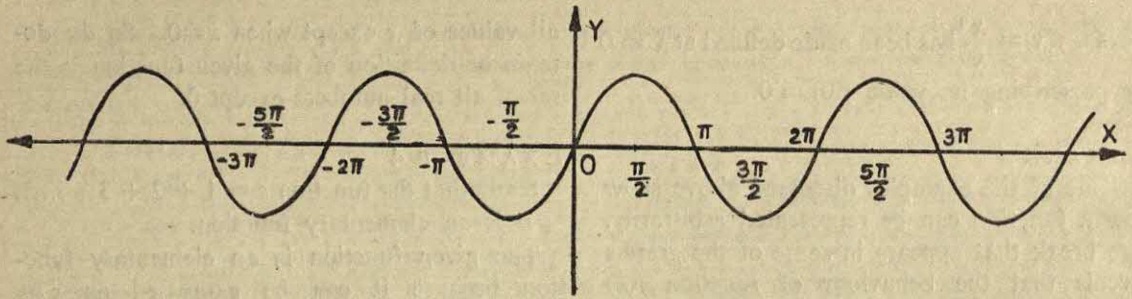

Fig. 1.9

EXAMPLE 8

If $f(\theta)=\tan\theta$, find $f(2\theta)$

$$\text{Here } f(2\theta)=\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2f(\theta)}{1-[f(\theta)]^2}$$

EXAMPLE 9

Construct the graph of the function $y=\sin x$. Here y is defined for all real values of x. The variation of y with x can be tabulated as:

TABLE 1-2

x	0	$\frac{\pi}{2}$	$-\frac{\pi}{2}$	π	$-\pi$	$\frac{3\pi}{2}$	$-\frac{3\pi}{2}$	etc
y	0	1	-1	0	0	-1	1	

If we now plot points (x,y), given in Table 1-2, in rectangular cartesian frame (x-y) and join them to cover intermediate points we have the graph as (Fig. 1.9).

Fig. 1.9 shows that the curve represented by the given function $y=\sin x$ has an infinite number of oscillations between the values $+1$ and -1. It is further to be noted that the graph has no break at any point.

EXAMPLE 10

Construct the graph of $y=x\sin\dfrac{1}{x}$.

Here y is not defined for $x=0$. Thus, the domain of definition of the function is the aggregate of all real numbers except 0.

The graph is shown in the figure which has a break at $x=0$, the corresponding point being missing on the graph. Near 0 on either side, the graph has an infinite number of oscillations with gradually diminishing amplitude. The graph is comprised between the lines $y=x$ and $y=-x$.

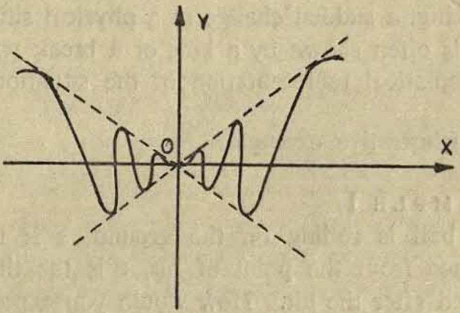

Fig. 1.10

EXAMPLE 11

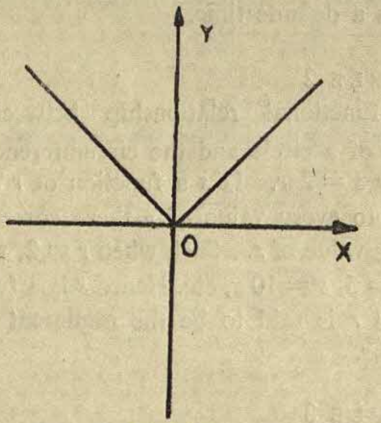

Fig. 1.11

Draw the graph of the function $y=|x|$ defined by

$$\begin{aligned} y &= x,\ x > 0 \\ &= 0,\ x = 0 \\ &= -x,\ x < 0 \end{aligned}$$

So the graph consists of the lines $y = x$ for all $x > 0$, $y = -x$ for all $x < 0$. These two lines meet at $x = 0$ since the value of $y = 0$ at $x = 0$. The lines have been drawn in an usual way as before.

Exercise 1.1

1. Find the functional relationship between the volume (v) and radius (r) of a solid sphere and state also the nature of the function that you establish.

2. The distance (by road) between the points A and B is 90 km. A motor car covered the first half of the distance from A to B at a speed of 0.6 km/min, the second at a speed of 0.9 km/min. If S (km) be the distance of the car from point A and t (min) the time for transit then express t as a function of S to represent the above situation.

3. If $\phi(\alpha) = \dfrac{1}{1+\text{Sin}^2\alpha}$,

find $\phi(0)$, $\phi\left(\dfrac{\pi}{2}\right)$, $\phi(\pi)$ and $\phi\left(\dfrac{\pi}{4}\right)$.

4. If $f(x) = \sqrt{x^2+1}$, find $f(1)$ and $f(\sqrt{3})$.

5. $\phi(x) = \log \dfrac{1-x}{1+x}$ verify the equality

$\phi(a) + \phi(b) = \phi\left(\dfrac{a+b}{1+ab}\right)$.

6. $f(x) = \log x$, $\phi(x) = \sqrt{x}$ write the expression $f[\phi(2)]$.

7. Find the domain of definition of following functions

(i) $y = \sqrt{1-x^2}$, (ii) $y = \log x$

(iii) $y = b^x\ (b > 0)$

(iv) $y = \sqrt{\sin 3x}$, (v) $y = \dfrac{x-2}{x^2-4}$,

(vi) $y = \sqrt{(5-x)(3x+6)}$

8. $f(x) = 2x^4 - 3x^3 - 5x^2 + 6x - 10$. Find

$\phi(x) = \frac{1}{2}[f(x) + f(-x)]$ and

$\psi(x) = \frac{1}{2}[f(x) - f(-x)]$

9. A function $f(x)$ defined in a symmetric region $-a < x < a$ is called **even** if $f(-x) = f(x)$ and **odd** if $f(-x) = -f(x)$. Determine which of the following functions are even and which are odd:

(i) $f(x) = \frac{1}{2}(a^x + a^{-x})$

(ii) $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$

(iii) $f(x) = \log \dfrac{1+x}{1-x}$

10. Prove that any function $f(x)$ defined in the interval $-l < x < l$ may be represented in the form of the sum of an even function and an odd function.

11. Determine the inverse of the functions: $y = 1 - 2^{-x}$ and $y = (x-2)(8-x)$

12. If $f(x)$ be the sum of n terms of an arithmetic progression, show that

$f(n+3) - 3f(n+2) + 3f(n+1) - f(n) = 0$

13. Show that if $f(x) = kx + b$ and the numbers x_1, x_2, x_3 are in A. P. then the numbers $f(x_1)$, $f(x_2)$ and $f(x_3)$ likewise form such a progression.

14. A function $f(x)$ is called **periodic** if there exists a positive number T (the period of the function) such that $f(x+T) = f(x)$ for all values of x within the domain of definition of $f(x)$.

Determine which of the following functions are periodic and for the periodic functions, find their (least) period T:

(i) $\sin x$, (ii) $\tan x$, (iii) $\sin nx$, (iv) $\sin 3x$,

(v) $\sin(\sqrt{x})$

15. $\phi(x) = \frac{1}{2}(a^x + a^{-x})$ and

$\psi(x) = \frac{1}{2}(a^x - a^{-x})$

Show that

$$\phi(x+y) = \phi(x)\,\phi(y) + \psi(x)\,\psi(y)$$

and $$\psi(x+y) = \phi(x)\,\psi(y) + \phi(y)\,\psi(x)$$

16. Find the inverse of the function

$$y = \begin{cases} x, & \text{if } x \leqslant 0 \\ x^2, & \text{if } x > 0 \end{cases}$$

17. Construct graphs of the following functions:

(i) $f(x) = \begin{cases} 1, & \text{if } x \text{ is a rational number} \\ 0, & \text{if } x \text{ is an irrational number.} \end{cases}$

(ii) $f(x) = \begin{cases} x^2, & x \neq 2 \\ 0, & x = 2 \end{cases}$

(iii) $f(x) = |x|, -3 \leqslant x \leqslant 3$

(iv) $f(x) = \dfrac{x-[x]}{x}$ where $[x]$ = greatest integer $\leqslant x$

(v) $f(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ 1, & x > 0 \end{cases}$

(vi) $f(x) = \dfrac{\text{Sin } x}{x}$

(vii) $f(x) = \begin{cases} -x, & -2 \leqslant x \leqslant 0 \\ x, & 0 \leqslant x < 2 \end{cases}$

(viii) $f(x) = x^2 \sin \dfrac{1}{x}$, $x \neq 0$

(ix) $f(x) = \dfrac{\sin^2 x}{x^2}$

(x) $f(x) = \cos\left(x - \dfrac{\pi}{3}\right)$

18. From the graph of $y = \cos x$ construct the graph of $y = \cos^{-1}x$. Show graphically why $\cos^{-1}x$ is a multiple-valued function.

19. Draw the graph of the function defined in the interval $(-1, 1)$ as follows:

$$f(x) = 1 + x \text{ for } -1 \leqslant x < 0$$
$$f(x) = 1 - 2x \text{ for } 0 \leqslant x < 1.$$

2 Limit and Continuity

IN Chapter One, we studied functions, their representations by means of graphs, and some of their properties. As we have observed in graphs of functions, there are some striking changes in their values owing to changes in values of their independent variables. In fact, behaviour of such functions at a point or at points close to a particular point or points is very important, specially on account of their use in physical or social or biological problems that deal with small changes in their respective entities. We shall study in this chapter behaviour of functions from this standpoint.

We make the independent variable x assume values on the both sides of the given point, say, $x = a$, but near to it. In other words, we make x gradually approach towards the point 'a', which we express as x **tends to** a. As x assumes these values in the close proximity of 'a' or in the neighbourhood of a, what would happen to $f(x)$? The answer is: $f(x)$ would go on assuming values. These values of $f(x)$ may tend to a certain constant, say, l. We would then say, $f(x)$ tends to the **limit** l. This leads us to an important concept, namely, **limit in differential calculus**.

Geometrically speaking, in order to study behaviour of a function near a particular point, it is necessary to observe changes in the values of the function on either side of the point.

2.1 Meaning of 'x tends to a'

Let us suppose that 'a' is a fixed point on OX and x assumes any of the set of numbers represented in a real number line OX on either side of 'a' and including 'a'. Hence x can approach 'a' both from right hand side and left hand side of 'a'. In other words x can approach 'a', remaining always greater than 'a' and also it can move towards 'a', remaining always less than 'a'.

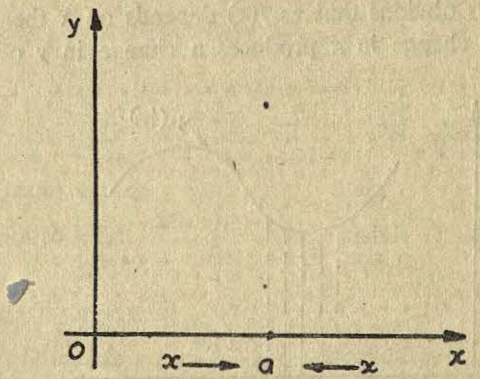

Fig. 2.1

Hence the phrase 'x tends to a' means that x approaches (or tends to) 'a' from either side of 'a' and can assume a value very near to 'a' (as near as it may be) but not exactly the value 'a'. In other words, x assumes successively values whose numerical difference from a, i.e. $|x-a|$, is gradually less and ultimately becomes very very small (as small as it can) when x is very very close to 'a'. 'x tends to a' is symbolically denoted as $x \rightarrow a$ or $Lt\ x = a$.

The notation corresopnding to 'x tends to

a from the right' is $x \longrightarrow a + 0$ or $x \longrightarrow a+$ while that corresponding to 'x tends to a from left' is $x \longrightarrow a - 0$ or $x \longrightarrow a-$. As a whole the limit $x \longrightarrow a$ includes both $x \longrightarrow a+$ and $x \longrightarrow a-$.

EXAMPLE 1

If x successively assumes values 0.9, 0.99, 0.999, 0.9999, etc. we say $x \longrightarrow 1-$. Again if x successively assumes values 1.1, 1.01, 1.001, 1.0001, etc. we say $x \longrightarrow 1+$. If the successive values are $1 + .1$, $1 - .1$, $1 + .01$, $1 - .01$, etc. we say $x \longrightarrow 1$.

2.2 Limit of a function

It is obvious that as $f(x)$ depends on x then every change in x produces a change in y or $f(x)$. We should, therefore, study how $f(x)$ changes gradually as x tends to a. These gradual changes in $f(x)$ suggest a value to which $f(x)$ may ultimately approach as x approaches very near to a. This particular value to which $f(x)$ can ultimately approach as x tends to a is called the limiting value of $f(x)$ as $x \longrightarrow a$ and is denoted by $\underset{x \longrightarrow a}{Lt}\, f(x) = l$ (any particular value).

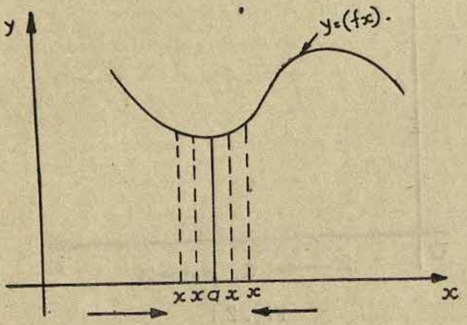

Fig. 2.2

$\underset{x \longrightarrow a}{Lt}\, f(x) = l$ means that $f(x)$ tends to the number l (called the limit l) when x approaches to the point $x = a$ both from left and from right of the point $x = a$.

Or, in other words, the number l is the limit of the function $f(x)$ as $x \longrightarrow a$, if the absolute value $| f(x) - l |$ is arbitrarily small when $| x - a |$ is sufficiently small.

REMARK

It is assumed that the function $f(x)$ is defined within some interval containing the point $x = a$ (at all points on the left and on the right of a) ; at the point $x = a$ itself, the function $f(x)$ is either defined or is not.

EXAMPLE 1

Consider the function $f(x) = 2x + 1$ (it is defined at all points). If $x = 2$ then $f(x) = 5$. As x approaches 2 (from the right and from the left) then $2x + 1$ tends to 5. The number 5 (equal to the value of the function at $x = 2$) is at the same time the limit of the function as $x \longrightarrow 2$:

$$\underset{x \to 2}{Lt}\quad 2x+1=5$$

EXAMPLE 2

Consider the function $f(x) = \dfrac{4x^2 - 1}{2x - 1}$ (it is defined at all points with the exception of $x = \frac{1}{2}$). If we have $x = 6$, then $f(x) = \dfrac{4.6^2 - 1}{2.6 - 1} = 13$. As x approaches 6 (from the right or from the left), the numerator $4x^2 - 1$ tends to 143 and the denominator tends to 11. The function as a whole tends to $\frac{143}{11} = 13$. The number 13 (equal to the value of the function at $x = 6$) is at the same time the limit of the function as $x \longrightarrow 6$:

$$\underset{x \to 6}{Lt}\ \frac{4x^2 - 1}{2x - 1} = 13$$

EXAMPLE 3

Consider the same function $f(x) = \dfrac{4x^2 - 1}{2x - 1}$ but take $x = \frac{1}{2}$. The function is not defined here (the formula yields the indeterminate form $\frac{0}{0}$). But the limit of the function exists as $x \longrightarrow \frac{1}{2}$. It is equal to 2.

Indeed, the expression $\dfrac{4x^2-1}{2x-1}$ is determinate only for $x=\frac{1}{2}$, but as x approaches $\frac{1}{2}$, it is quite determinate and is always equal to $2x+1$. This expression tends to the number 2. Hence

$$\underset{x\to\frac{1}{2}}{Lt}\ \frac{4x^2-1}{2x-1}=2$$

The graph of the function $y=\dfrac{4x^2-1}{2x-1}$ is the straight line UV (Fig. 2.3) omitting the point A $(\frac{1}{2}, 2)$. The graph of the function $y=2x+1$ is the same straight line UV taken in its entirety.

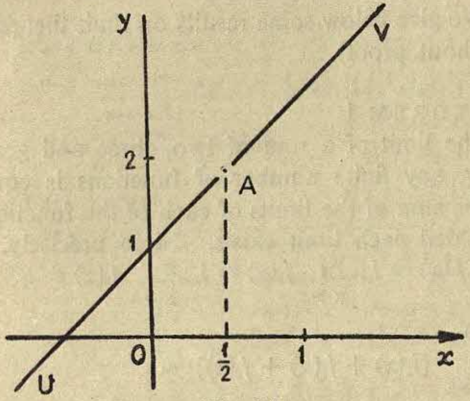

Fig. 2.3

EXAMPLE 4

The function $f(x)=\cos \pi/x$ (it is defined at all points except $x=0$) does not have a limit as $x \longrightarrow 0$. This is evident from Fig. 2.4.

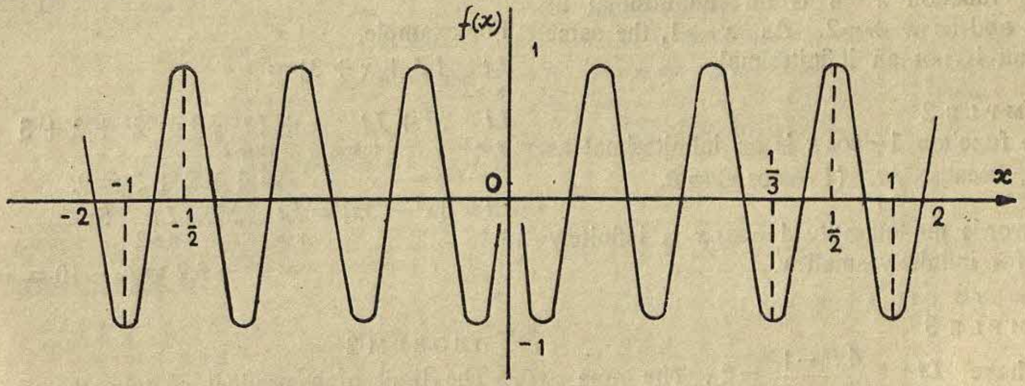

Fig. 2,4

When the abscissa approaches zero, the ordinate does not approach anything (the point of the graph performs infinite oscillations with a constant amplitude).

2.3 Illustrative Examples

Let us now go through some illustrations to get an idea of the existence of limits of functions :

EXAMPLE 1

Does $\underset{x\to 1}{Lt}\ \dfrac{x^2-1}{x-1}$ exist ?

Here $\underset{x\to 1+0}{Lt}\ \dfrac{x^2-1}{x-1}=\underset{x\to 1+0}{Lt}\ (x+1)=2$

and $\underset{x\to 1-0}{Lt}\ \dfrac{x^2-1}{x-1}=\underset{x\to 1-0}{Lt}\ (x+1)=2$

($\because$ $x \longrightarrow 1$ but not equal to 1).

$\therefore$ $\underset{x\to 1+0}{Lt}\ \dfrac{x^2-1}{x-1}=\underset{x\to 1+0}{Lt}\ \dfrac{x^2-1}{x-1}=2$

(finite value)

Hence $\underset{x\to 1}{Lt}\ \dfrac{x^2-1}{x-1}$ exists.

EXAMPLE 2

Does $\underset{x\to 0}{Lt}\ \dfrac{x-|x|}{x}$ exists ?

Here, Let

$f(x)=\dfrac{x-|x|}{x}$ and $|x|=x$ when $x>0$
$=0 \quad x=0$
$=-x \quad x<0$

We have, then

$$\underset{x\to 0+}{Lt}\ f(x) = \underset{x\to 0+}{Lt}\ \frac{x-|x|}{x}$$

$$= \underset{x\to 0+}{Lt}\ \frac{x-x}{x} = 0$$

and $$\underset{x\to 0-}{Lt}\ f(x) = \underset{x\to 0-}{Lt}\ \frac{x-|x|}{x}$$

$$= \underset{x\to 0-}{Lt}\ \frac{x+x}{x} = 2$$

$$\therefore\ \underset{x\to 0+}{Lt}\ f(x) \neq \underset{x\to 0-}{Lt}\ f(x)$$

Hence $\underset{x\to 0}{Lt}\ f(x)$ does not exist.

REMARK

If $\underset{x\to a-0}{Lt}\ f(x) = l_1$ and $\underset{x\to a+0}{Lt}\ f(x) = l_2$ and $l_1 = l_2 = l$ then we say that $\underset{x\to a}{Lt}\ f(x)$ exists and is equal to l.

2.4 Some necessary ideas and definitions

(i) The limit of a constant quantity C is the quantity C itself.

This definition is introduced so that basic theorems on limits should hold true in all cases without exception. It agrees with definition of 2.3 ($|c-c| = 0$ when $|x-a|$ is very small).

(ii) An **infinitesimal** is a quantity whose limit is equal to zero.

EXAMPLE 1

The function x^2-4 is an infinitesimal as $x\longrightarrow 2$ and as $x\longrightarrow -2$. As $x\longrightarrow 1$, the same function is not an infinitesimal.

EXAMPLE 2

The funciton $1-\cos x$ is an infinitesimal as $x\longrightarrow 0$, because $\underset{x\to 0}{Lt}\ (1-\cos x) = 0$.

In words the quantity $1-\cos x$ is infinitely small for infinitely small x.

EXAMPLE 3

We have $\underset{x\to\frac{1}{2}}{Lt}\ \frac{4x^2-1}{2x-1} = 2$. The same fact may be expressed as the quantity $\frac{4x^2-1}{2x-1}-2$ which is an infinitesimal.

REMARKS

1. The statements 'the number l is the limit of the variable y' and 'the difference $y-l$ is an infinitesimal' are equivalent.

2. Of all constant quantities, only zero is an infinitesimal.

2.5 Basic Limit theorems

We give below some results on limit theorems (without proof).

THEOREM 1

The limit of a sum of two, three and generally, any finite number of functions is equal to the sum of the limits of each of the functions provided each limit exists. More precisely, if $\underset{x\to a}{Lt}\ f_1(x) = l_1$, $\underset{x\to a}{Lt}\ f_2(x) = l_2$, $\underset{x\to a}{Lt}\ f_3(x) = l_3$.

Then

$$\underset{x\to a}{Lt}\ [f_1(x) + f_2(x) + f_3(x)] =$$

$$\underset{x\to a}{Lt}\ f_1(x) + \underset{x\to a}{Lt}\ f_2(x) + \underset{x\to a}{Lt}\ f_3(x)$$

$$= l_1 + l_2 + l_3$$

THEOREM 1a

$$\underset{x\to a}{Lt}\ [f_1(x) - f_2(x)] = \underset{x\to a}{Lt}\ f_1(x) - \underset{x\to a}{Lt}\ f_2(x) = l_1 - l_2$$

For example,

$$\underset{x\to 2}{Lt}\ [x^2 + x + 3] =$$

$$\underset{x\to 2}{Lt}\ x^2 + \underset{x\to 2}{Lt}\ x + \underset{x\to 2}{Lt}\ 3 = 2^2 + 2 + 3$$

$$= 4 + 2 + 3 = 9.$$

$$\underset{x\to 2}{Lt}\ [x^2 - 5x] = \underset{x\to 2}{Lt}\ x^2 - \underset{x\to 2}{Lt}\ 5x$$

$$= 2^2 - 5.2 = 4 - 10 = -6$$

THEOREM 2

The limit of a product of two, three and, generally, any finite number of functions is equal to the product of their limits:

i.e. $\underset{x\to a}{Lt}\ [f_1(x)\,f_2(x)\,f_3(x)]$

$$= \underset{x\to a}{Lt}\ f_1(x) \times \underset{x\to a}{Lt}\ f_2(x) \times \underset{x\to 0}{Lt}\ f_3(x)$$

$$= l_1 \times l_2 \times l_3$$

For example,

(i) $\underset{x\to 0}{Lt}\ [x\cos x] = \underset{x\to 0}{Lt}\ x \times \underset{x\to 0}{Lt}\ \cos x$

$$= 0 \times 1 = 1$$

(ii) $\underset{x\to 0}{Lt}\ (ax+b)(cx+d)$

$= \underset{x\to 0}{Lt}\ (ax+b) \times \underset{x\to 0}{Lt}\ (cx+d) = b \times d = bd$

THEOREM 2a

$$\underset{x\to a}{Lt}\left\{ c f_1(x) \right\} = c\ \underset{x\to a}{Lt}\ f_1(x) = c\, l_1$$

where c is a constant

For example,

$\underset{x\to 1}{Lt}\ (5x^2 + 3x + 2)$

$$= \underset{x\to 1}{Lt}\ 5x^2 + \underset{x\to 1}{Lt}\ 3x + \underset{x\to 1}{Lt}\ 2$$

$$= 5\underset{x\to 1}{Lt}\ x^2 + 3\underset{x\to 1}{Lt}\ x + \underset{x\to 1}{Lt}\ 2$$

$$= 5.1 + 3.1 + 2 = 10$$

THEOREM 3

The limit of a quotient of two functions is equal to the quotient of the limits of the functions if the limit of the divisor is not equal to zero:

$$\underset{x\to a}{Lt}\ \frac{f_1(x)}{f_2(x)} = \frac{\underset{x\to a}{Lt}\ f_1(x)}{\underset{x\to a}{Lt}\ f_2(x)} = \frac{l_1}{l_2}$$

provided $l_2 \neq 0$

For example,

$$\underset{x\to 5}{Lt}\ \frac{x+4}{x-2} = \underset{x\to 5}{Lt}\ (x+4) \Big/ \underset{x\to 5}{Lt}\ (x-2)$$

$$= \tfrac{9}{3} = 3$$

THEOREM 4

$$\underset{x\to a}{Lt}\ F\left\{ f(x) \right\} = F\left\{ \underset{x\to a}{Lt}\ (fx) \right\} = F(l)$$

where F stands for a symbol for an operator.

For example,

$$\underset{x\to a}{Lt}\ \log f(x) = \log \left\{ \underset{x\to a}{Lt}\ f(x) \right\}$$

$$= \log l, \text{ provided } l > 0.$$

$$\underset{x\to 0}{Lt}\ e^{\sin x} = e^{\underset{x\to 0}{Lt}\ \sin x} = e^0 = 1$$

REMARK

We shall see that we can compute limits of functions with the help of above results and with the help of some well known limits, known as **standard limits.**

2.6 Some standard limits

(i) $\underset{x\to 0}{Lt}\ \dfrac{\sin x}{x} = 1$ if x is the radian measure of the angle.

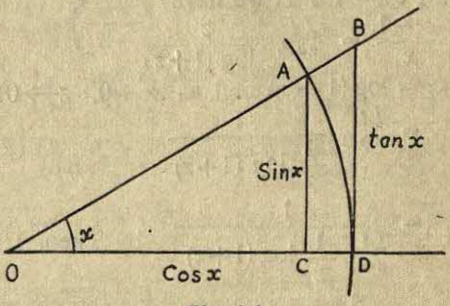

Fig. 2.5

Let us construct a circle with centre at O and radius $OA = OD = 1$ as in Fig. 2.5. We choose point B on OA extended and point C on OD so that lines BD and AC are perpendicular to OD.

It is evident that Area of triangle $OAC <$ Area of sector $OAD <$ Area of triangle OBD, i.e. $\frac{1}{2}\sin x\ \cos x < \frac{1}{2}x < \frac{1}{2}\tan x$.

Dividing by $\frac{1}{2}\sin x$,

$$\cos x < \frac{x}{\sin x} < \frac{1}{\cos x}$$

or, $$\cos x < \frac{\sin x}{x} < \frac{1}{\cos x}$$

As $x \to 0$, $\cos x \to 1$ and we have $1 < \dfrac{\sin x}{x} < 1$ as $x \to 0$.

This means $\underset{x\to 0}{Lt}\ \frac{\sin x}{x} = 1$

(ii) $\underset{x\to 0}{Lt}\ (1+x)^{\frac{1}{x}} = e$ where $e = 2.71828$ (approx)

The proof of it is not given here.

(iii) $\underset{x\to 0}{Lt}\ \frac{1}{x}\log(1+x) = 1$

From the standard limit (ii) and using Theorem 4 we have

$$\underset{x\to 0}{Lt}\ \frac{1}{x}\log(1+x) = \underset{x\to 0}{Lt}\ \log(1+x)^{\frac{1}{x}}$$

$$= \log\left\{\underset{x\to 0}{Lt}\ (1+x)^{\frac{1}{x}}\right\} = \log e = 1$$

(iv) $\underset{x\to 0}{Lt}\ \frac{e^x-1}{x} = 1.$

Putting $e^x = 1 + z$, we have

$$\underset{x\to 0}{Lt}\ \frac{e^x-1}{x} = \underset{z\to 0}{Lt}\ \frac{z}{\log(1+z)}$$

($\because$ $x = \log(1+z)$ and as $x \to 0$, $z \to 0$)

$$= \underset{z\to 0}{Lt}\ \frac{1}{\frac{1}{z}\log(1+z)}$$

$$= \frac{1}{\underset{z\to 0}{Lt}\ \frac{1}{z}\log(1+z)}$$ by Theo. III

$$= \frac{1}{1} = 1$$ by standard result (iii).

2.7 Illustrative examples

EXAMPLE 1

Find the right-hand and left-hand limit of the function $f(x) = \tan^{-1}\frac{1}{x}$ as $x \to 0$.

We have,

$$\underset{x\to 0+}{Lt}\ f(x) = \underset{x\to 0+}{Lt}\ \tan^{-1}\frac{1}{x} = \frac{\pi}{2}$$

$$\underset{x\to 0-}{Lt}\ f(x) = \underset{x\to 0-}{Lt}\ \tan^{-1}\frac{1}{x} = -\frac{\pi}{2}$$

Obviously, the function $f(x)$ in this case has no limit as $x \to 0$.

EXAMPLE 2

Evaluate $\underset{x\to 2}{Lt}\ \frac{x^2-4}{x^2-3x+2}$

We have,

$$\underset{x\to 2}{Lt}\ \frac{x^2-4}{x^2-3x+2} = \underset{x\to 2}{Lt}\ \frac{(x-2)(x+2)}{(x-2)(x-1)}$$

$$= \underset{x\to 2}{Lt}\ \frac{x+2}{x-1} = 4$$

EXAMPLE 3

Find $\underset{x\to 0}{Lt}\ \frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}$

Putting $1 + x = y^6$, we have as $x \to 0$, $y \to 1$ and

$$\underset{x\to 0}{Lt}\ \frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1} = \underset{y\to 1}{Lt}\ \frac{y^3-1}{y^2-1}$$

$$= \underset{y\to 1}{Lt}\ \frac{y^2+y+1}{y+1} = \frac{3}{2}$$

EXAMPLE 4

$$\underset{x\to a}{Lt}\ \frac{\sqrt{x}-\sqrt{a}}{x-a} = \underset{x\to a}{Lt}\ \frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})}$$

$$= \underset{x\to a}{Lt}\ \frac{1}{\sqrt{x}+\sqrt{a}} = \frac{1}{2\sqrt{a}},\ (a>0)$$

EXAMPLE 5

$$\underset{x\to 0}{Lt}\ \frac{\sin 5x}{x} = \underset{x\to 0}{Lt}\left(\frac{\sin 5x}{5x}.5\right) = 1.5 = 5$$

EXAMPLE 6

Find $\underset{x\to 0}{Lt}\left(\frac{\sin 2x}{x}\right)^{1+x}$

Here, $\underset{x\to 0}{Lt}\left(\frac{\sin 2x}{x}\right) = \underset{x\to 0}{Lt}\ \frac{\sin 2x}{2x}.2 = 2$

and $\underset{x\to 0}{Lt}\ (1+x) = 1$

Hence $\underset{x\to 0}{Lt}\left(\frac{\sin 2x}{x}\right)^{1+x} = 2^1 = 2$

EXAMPLE 7

Prove that $\underset{x\to a}{Lt}\ \frac{x^n-a^n}{x-a} = na^{n-1}$ when n is a $+ve$ integer.

By actual division, we have

$$\frac{x^n-a^n}{x-a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots\ldots + a^{n-1}$$

$$\therefore \underset{x \to a}{Lt} \frac{x^n - a^n}{x - a} = \underset{x \to a}{Lt} (x^{n-1} + x^{n-2} a + x^{n-3} a^2 + \ldots\ldots + a^{n-1}) = na^{n-1}$$

EXAMPLE 8

Prove that $\underset{x \to 1}{Lt} \dfrac{\sqrt{x}}{x+1}$ does not exist.

We have $\underset{x \to 1+}{Lt} \dfrac{\sqrt{x}}{x+1} = \dfrac{1}{2}$

and $\underset{x \to 1-}{Lt} \dfrac{\sqrt{x}}{x+1} = \dfrac{1}{2}$

Hence $\underset{x \to 1+}{Lt} \dfrac{\sqrt{x}}{x+1} = \underset{x \to 1-}{Lt} \dfrac{\sqrt{x}}{x+1} = \dfrac{1}{2}$

So $\underset{x \to 1}{Lt} \dfrac{\sqrt{x}}{x+1}$ exists and is equal to ½.

EXAMPLE 9

The function $f(x)$ is defined as

$$\begin{aligned} f(x) &= 3x-1 \quad x < 0 \\ &= 0 \qquad\quad x = 0 \\ &= 2x+5 \quad x > 0 \end{aligned}$$

Does $\underset{x \to 0}{Lt} f(x)$ exist?

We have

$$\underset{x \to 0+}{Lt} f(x) = \underset{x \to 0+}{Lt} 2x + 5 = 5$$

$$\underset{x \to 0-}{Lt} f(x) = \underset{x \to 0-}{Lt} 3x - 1 = -1$$

$\therefore \underset{x \to 0+}{Lt} f(x) \neq \underset{x \to 0-}{Lt} f(x)$. So $\underset{x \to 0}{Lt} f(x)$ does not exist.

Exercise 2.1

1. Evaluate the following limits:

(i) $\underset{x \to 1}{Lt} \dfrac{x^3+1}{x^2+1}$ (ii) $\underset{h \to 0}{Lt} \dfrac{(x+h)^3 - x^3}{h}$

(iii) $\underset{x \to 1}{Lt} \left(\dfrac{1}{1-x} - \dfrac{3}{1-x^3}\right)$ (iv) $\underset{x \to 1}{Lt} \dfrac{\sqrt[3]{x} - 1}{\sqrt[4]{x} - 1}$

(v) $\underset{x \to 1}{Lt} \dfrac{\sqrt[3]{x^2} - 2\sqrt[3]{x} + 1}{(x-1)^2}$ (vi) $\underset{x \to 7}{Lt} \dfrac{2 - \sqrt{x-3}}{x^2 - 49}$

(vii) $\underset{h \to 0}{Lt} \dfrac{\sqrt{x+h} - \sqrt{x}}{h}$

(viii) $\underset{x \to 3}{Lt} \dfrac{\sqrt{x^2-2x+6} - \sqrt{x^2+2x-6}}{x^2-4x+3}$

(ix) $\underset{x \to 2}{Lt} \dfrac{\sin x}{x}$ (x) $\underset{x \to 0}{Lt} x \sin \dfrac{1}{x}$

(xi) $\underset{x \to 1}{Lt} (1-x) \tan \dfrac{\pi x}{2}$ (xii) $\underset{x \to 0}{Lt} \dfrac{\sin 5x}{\sin 2x}$

(xiii) $\underset{x \to \pi}{Lt} \dfrac{1 - \sin \frac{x}{2}}{\pi - x}$ (xiv) $\underset{x \to 0}{Lt} \dfrac{\pi - \cos x}{x^2}$

(xv) $\underset{x \to a}{Lt} \dfrac{\sin x - \sin a}{x - a}$

(xvi) $\underset{x \to 0}{Lt} \dfrac{\cos mx - \cos nx}{x^2}$

(xvii) $\underset{x \to a}{Lt} \dfrac{1 - \sqrt{\cos x}}{x^2}$

(xviii) $\underset{x \to 0}{Lt} \dfrac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x}$

(xix) $\underset{x \to 0}{Lt} \left(\dfrac{2+x}{3-x}\right)^x$ (xx) $\underset{x \to 0}{Lt} \dfrac{\log(1+10x)}{x}$

(xxi) $\underset{x \to 0}{Lt} \dfrac{e^{ax} - e^{bx}}{x}$ (xxii) $\underset{x \to 0}{Lt} \dfrac{(\cos x)}{x^2}$

(xxiii) $\underset{x \to 0}{Lt} \dfrac{e^x - 1}{x}$ (xxiv) $\underset{x \to 0}{Lt} \dfrac{a^x - 1}{x}$

(xxv) $\underset{x \to 0}{Lt} \dfrac{1 - e^{-x}}{\sin x}$ (xxvi) $\underset{h \to 0}{Lt} \dfrac{\sqrt[3]{8+h} - 2}{h}$

2. If $f(x) = \begin{cases} 3x-1, & x < 0 \\ 0 \quad, & x = 0 \\ 2x+5, & x > 0 \end{cases}$

Evaluate

$\underset{h \to 0+}{Lt} \dfrac{f(h) - f(0+)}{h}$ and $\underset{h \to 0-}{Lt} \dfrac{f(h) - f(0-)}{h}$.

Construct also the graph of the function.

2.8 The increment of a variable quantity

If a variable x assumes the value $x = x_1$ and then $x = x_2$, then the difference $x_2 - x_1$ is called the **increment of** x. The increment may be positive, negative, or zero. The increment is denoted by the Greek letter $\triangle$ (delta) (the symbol $\triangle x$ reads 'delta x'). It denotes the change in x, 'increment of the quantity x'; we have

$$\triangle x = x_2 - x_1$$

REMARK

The increment of a constant is zero.

REMARK

The initial value of $x = 3$, the increment in

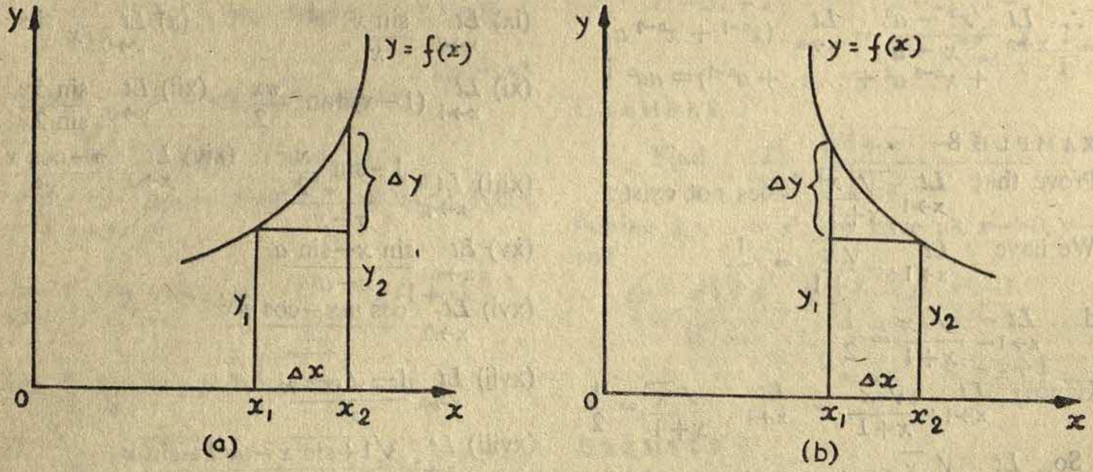

Fig. 2.6

x, i.e. $\triangle x = -2$, determine the corresponding increment $\triangle y$, of the function $y = x^2$.

Here since $x_1 = 3$ and $x_2 - x_1 = -2$, it follows that $x_2 = 1$. The function $y = x^2$ first takes on the value $y_1 = 3^2 = 9$ and then $y_2 = 1^2 = 1$. The increment of the function is $\triangle y = y_2 - y_1 = 1 - 9 = -8$. The positive and negative increments are shown in Figure 2.6(a) and (b).

$$\triangle y = y_2 - y_1 > 0 \text{ for } y_2 < y_1$$
$$\text{and } \triangle x = x_2 - x_1 > 0 \text{ for } x_2 > x_1$$
$$\triangle y = y_2 - y_1 < 0 \text{ for } y_2 < y_1$$
$$\triangle x = x_2 - x_1 > 0 \text{ for } x_2 > x_1$$

2.9 The continuity of a function at a point

Let the function $y = f(x)$ be defined for some value $x = a$ and in some neighbourhood (either side) of $x = a$ and let its value at $x = a$ be $f(a)$. Now if x receives some positive or negative increments $\triangle x$ and assumes the value $x = a + \triangle x$, then the function y too will receive an increment $\triangle y$. The new increased value of the function will be $y + \triangle y = f(a + \triangle x)$ (Fig. 2.7). The increment of the function then becomes $\triangle y = f(a + \triangle x) - f(a)$.

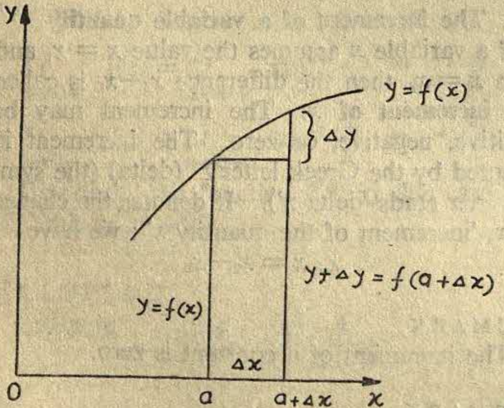

Fig. 2.7

The function $y = f(x)$ is called **continuous** for the value $x = a$ (or at the point $x = a$) if it is defined on either side of a including the point a and if

$$\underset{\triangle x \to 0}{Lt} \triangle y = 0$$

$$\text{or, } \underset{\triangle x \to 0}{Lt} [f(a + \triangle x) - f(a)] = 0$$

$$\text{or, } \underset{\triangle x \to 0}{Lt} f(a + \triangle x) = f(a) \qquad (1)$$

$$\text{or, } \underset{\triangle x \to 0+}{Lt} f(a + \triangle x) = \underset{\triangle x \to 0-}{Lt} f(a + \triangle x) = f(a)$$

$$\text{or, } \underset{x \to a+}{Lt} f(x) = \underset{x \to a-}{Lt} f(x) = f(a)$$

In geometrical terms, the continuity of a function at a given point signifies that the difference of the ordinates of the graph of the function $y = f(x)$ at the points $a + \triangle x$ and a will, in absolute magnitude, be arbitrarily small, provided $|\triangle x|$ is sufficiently small.

To illustrate that the function $y = x^2$ is continuous at an arbitrary point a, we have at

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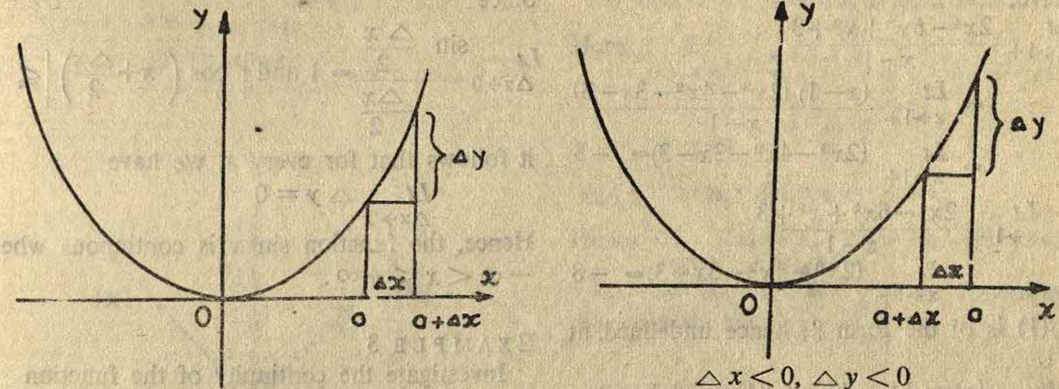


$\triangle x > 0,\ \triangle y > 0$
(a)

$\triangle x < 0,\ \triangle y < 0$
(b)

Fig. 2.8

$x = a,\ y = a^2$ and at $x + \triangle x,\ y + \triangle y = (a + \triangle x)^2$

$$\therefore\quad \triangle y = (a + \triangle x)^2 - a^2 = 2a \triangle x + (\triangle x)^2$$

Hence

$$\underset{\triangle x \to 0}{Lt} \triangle y = \underset{\triangle x \to 0}{Lt} [2a \triangle x + (\triangle x)^2] = 2a$$

for any way that $\triangle x$ may approach zero [Fig. 2.8(a) and (b)].

Further, to prove that the function $y = \sin x$ is continuous at an arbitrary point x, we have

$$y = \sin x,\ y + \triangle y = \sin (x + \triangle x)$$

$$\therefore\quad \triangle y = \sin (x + \triangle x) - \sin x$$

$$= 2 \sin \frac{\triangle x}{2} \cos \left(x + \frac{\triangle x}{2}\right)$$

$$\therefore\quad \underset{\triangle x \to 0}{Lt} \triangle y$$

$$= 2 \underset{\triangle x \to 0}{Lt} \left[\sin \frac{\triangle x}{2} \cos \left(x + \frac{\triangle x}{2}\right)\right]$$

$$= 2 \times 0 \times \cos x = 0$$

The above discussion suggests that the function $y = f(x)$ is continuous at a point $x = a$ if $\underset{x \to a}{Lt} f(x)$ exists and is equal to $f(a)$. Symbolically written, $f(x)$ is said to be continuous at $x = a$, if

$$\underset{x \to a+}{Lt} f(x) = \underset{x \to a-}{Lt} f(x) = f(a) \qquad (2)$$

REMARKS

1. If even one of the conditions (2) is violated the function is called discontinuous at the point $x = a$.

2. If a function $f(x)$ is continuous for every value of x in its domain (interval) of definition, it is said to be continuous throughout the domain.

3. Every elementary function is continuous at each point at which it is defined.

The condition for continuity (2) may be written thus:

$$\underset{\triangle x \to 0}{Lt} f(a + \triangle x) = f(a)$$

or, $\underset{x \to 0}{Lt} f(x) = f(a)$ but $a = \underset{x \to a}{Lt} x$

Consequently, $\underset{x \to a}{Lt} f(x) = f\left(\underset{x \to a}{Lt} x\right)$

In other words, in order to find the limit of a continuous function as $x \to a$ it is sufficient to substitute into the expression of the function the value of the argument, a, in place of the argument x, such as $\underset{x \to 3}{Lt} x^2 = 3^2 = 9$.

2.10 Illustrative examples

EXAMPLE 1

Is the function $f(x) = \dfrac{2x^4 - 6x^3 + x^2 + 3}{x - 1}$ continuous at $x = 1$?

We have,

$$\underset{x\to 1+}{Lt}\ \frac{2x^4-6x^3+x^2+3}{x-1}$$

$$=\underset{x\to 1+}{Lt}\ \frac{(x-1)(2x^3-4x^2-3x-3)}{x-1}$$

$$=\underset{x\to 1+}{Lt}\ (2x^3-4x^2-3x-3)=-8$$

and $\underset{x\to 1-}{Lt}\ \dfrac{2x^4-6x^3+x^2+3}{x-1}$

$$=\underset{x\to 1-}{Lt}\ (2x^3-4x^2-3x-3)=-8$$

But $f(1)$ is of the form $\frac{0}{0}$, hence undefined at $x=1$.

$\therefore\ \underset{x\to 1+}{Lt}\ f(x)=\underset{x\to 1-}{Lt}\ f(x)\neq f(1)$ (undefined)

Hence $f(x)$ is not continuous at $x=1$.

EXAMPLE 2

Prove that the function $y=\sin x$ is continuous at $x=0$. Prove also that $\sin x$ is continuous for every value of the argument x.

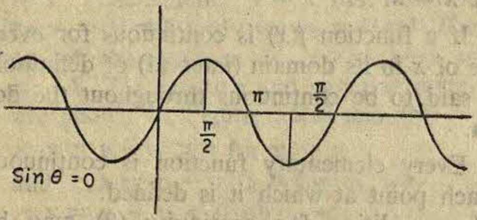


Fig. 2.9

(i) Here $\underset{x\to 0+}{Lt}\ \sin x=0\ \underset{x\to 0-}{Lt}\ \sin x=0$

and $\sin 0=0$.

$\therefore$ $\sin x$ is continuous at $x=0$.

(ii) Any function $f(x)$ is continuous at the point ξ if (and only if) at this point to an infinitesimal increment in ξ there corresponds an infinitesimal increment in the function, i.e.

$$\underset{\Delta\xi\to 0}{Lt}\ \Delta f(\xi)=\underset{\Delta\xi\to 0}{Lt}\ [f(\xi+\Delta\xi)-f(\xi)]=0$$

We have now in the problem

$$\triangle y=\sin(x+\triangle x)-\sin x=2\sin\frac{\triangle x}{2}\cos\left(x+\frac{\triangle x}{2}\right)$$

$$=\frac{\sin\frac{\triangle x}{2}}{\sin\frac{\triangle x}{2}}\cos\left(n+\frac{\triangle x}{2}\right)\cdot\triangle x$$

Since

$$\underset{\Delta x\to 0}{Lt}\ \frac{\sin\frac{\triangle x}{2}}{\frac{\triangle x}{2}}=1 \text{ and } \left|\cos\left(x+\frac{\triangle x}{2}\right)\right|\leqslant 1$$

it follows that for every x, we have

$$\underset{\Delta x\to 0}{Lt}\ \triangle y=0$$

Hence, the function $\sin x$ is continuous when $-\infty<x<+\infty$.

EXAMPLE 3

Investigate the continuity of the function

$$f(x)=\begin{cases}\sin\pi x, & 0<x<1\\ \log x, & 1<x<2\end{cases}$$

at $x=1$. Find also for what value of $f(1)$, the function is continuous at $x=1$.

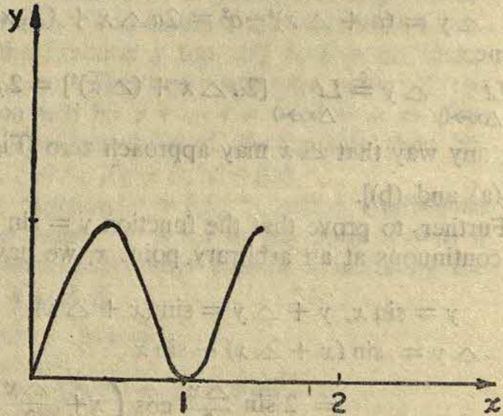


Fig. 2.10

Here $\underset{x\to 1+}{Lt}\ f(x)=\underset{x\to 1+}{Lt}\ \log x=0$

and $\underset{x\to 1-}{Lt}\ f(x)=\underset{x\to 1-}{Lt}\ \sin\pi x=0$

But $f(1)$ is not defined (since it is not known from definition of $f(x)$).

$\therefore\ \underset{x\to 1+}{Lt}\ f(x)=\underset{x\to 1-}{Lt}\ f(x)\neq f(1)$

Hence $f(x)$ is not continuous at $x=1$.

Further if we choose $f(1)=0$, then all the criteria for continuity of $f(x)$ at $x=1$ are being satisfied. Hence the function will be continuous at $x=1$ if $f(1)=0$.

EXAMPLE 4

Prove that $f(x) = |x|$ is continuous at $x = 0$.

We know

$$f(x) = |x| = x,\ x > 0$$
$$= 0,\ x = 0$$
$$= -x,\ x < 0$$

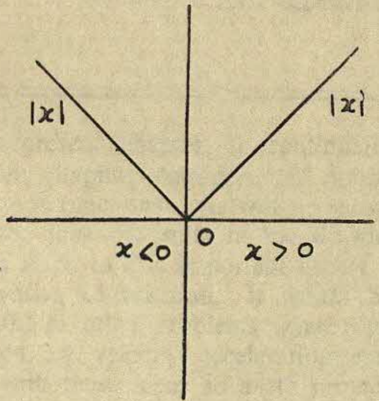


Fig. 2.11

Therefore

$$\underset{x \to 0+}{Lt} |x| = \underset{x \to 0+}{Lt} x = 0$$
$$\underset{x \to 0-}{Lt} |x| = \underset{x \to 0-}{Lt} (-x) = 0$$

and $|0| = 0$

$\therefore$ $f(x) = |x|$ is continuous at $x = 0$.

EXAMPLE 5

Is $f(x) = \dfrac{1}{x-3}$ continuous at $x = 2$?

To examine this, we find

$$\underset{x \to 2+0}{Lt} f(x) = \underset{x \to 2+0}{Lt} \frac{1}{x-3} = -1$$
$$\underset{x \to 2-0}{Lt} f(x) = \underset{x \to 2-}{Lt} \frac{1}{x-3} = -1$$

and $f(2) = -1$

$\therefore$ $\underset{x \to 2}{Lt} f(x)$ exists and is equal to $f(2)$ [=1]

$\therefore$ $f(x)$ is continuous at $x = 2$.

EXAMPLE 6

Is $f(x) = 2^{\frac{1}{x}}$ continuous at $x = 0$?

Here, $\underset{x \to 0+}{Lt} 2^{\frac{1}{x}} = \infty$

$\underset{x \to 0-}{Lt} 2^{\frac{1}{x}} = 0$

and $2^{\frac{1}{x}}$ is not defined at $x = 0$.

Hence $\underset{x \to 0+}{Lt} f(x) \neq \underset{x \to 0-}{Lt} f(x)$ and $f(0)$ is undefined. So the function $f(x)$ is discontinuous (not continuous) at $x = 0$.

Exercise 2.2

1. Prove that $f(x) = \dfrac{1}{x}$ is continuous at $a = 2$

2. Investigate the continuity of each of the following functions:

(i) $f(x) = \dfrac{\sin x}{x}, x \neq 0$; $f(0) = 0$, $x = 0$

(ii) $f(x) = x - |x|$, $x = 0$

(iii) $f(x) = \dfrac{x^3 - 8}{x^2 - 4}$, $x \neq 2$; $f(2) = 3$, $x = 2$.

3. If $[x] =$ greatest integer $\leqslant x$, investigate the continuity of $f(x) = x - [x]$ in the interval $1 \leqslant x \leqslant 2$.

4. Find the points of discontinuity of each of the following functions:

(i) $f(x) = \dfrac{x}{(x-2)(x-4)}$

(ii) $f(x) = \sqrt{(x-3)(x-6)}$, $3 \leqslant x \leqslant 6$

(iii) $f(x) = x^2 \sin \dfrac{1}{x}$, $x \neq 0$; $f(0) = 0$

(iv) $f(x) = \dfrac{x^3 + 2x + 5}{x^2 - 8x + 12}$.

5. Draw the graph of the function $f(x) = |x|$ and justify why it is continuous at $x = 0$.

6. $y = \begin{cases} x^2 \text{ for } x \leqslant 3 \\ 2x + 1 \text{ for } x > 3 \end{cases}$

Plot the graph of this function and explain geometrically the nature of the function.

7. The function $f(x)$ in the following is not defined for $x = 0$. Define, if possible, $f(0)$ so that $f(x)$ is continuous for $x = 0$.

(i) $f(x) = \dfrac{(1+x)^n - 1}{x}$ (n, +ve integer)

(ii) $f(x) = \dfrac{1-\cos x}{x^2}$,

(iii) $f(x) = \dfrac{\log(1+x) - \log(1-x)}{x}$

(iv) $f(x) = \dfrac{e^x - e^{-x}}{x}$

(v) $f(x) = x^2 \sin \dfrac{1}{x}$

(vi) $f(x) = x \cot x$

3 Derivatives

THE present chapter, in continuation of the earlier chapter, considers an important property of functions. As will be shown presently, the idea of limits in the drawing of tangents leads to this important aspect, called the **derivative of function.** It would likewise be helpful in other problems where rates are involved, e.g. velocity, acceleration, etc. We begin with these ideas so as to provide a background to the mathematical definition of derivatives.

3.1 Tangents to curves

Let $y = f(x)$ represent a curve. Let us seek to draw a tangent to the curve at any point say, $P(x, y)$. By definition, to have the tangent at P we need to find out the limiting position of a secant PQ (Q, a point close to P). In other words, in the language of limits, as *Q approaches P* (symbolically $Q \longrightarrow P$), the position of the secant in the limit will become the tangent at P. Let the coordinates of Q be $(x + \triangle x, \quad y + \triangle y)$ where $y + \triangle y = f(x + \triangle x)$. Then $PN = \triangle x$, $QN = \triangle y$, $\frac{QN}{PN}\left(= \frac{\triangle y}{\triangle x}\right)$ is, as you know, the gradient of PQ in coordinate geometry. What is the gradient of the tangent at P? As $Q \longrightarrow P$, i.e. $\triangle x \longrightarrow 0$, $\frac{\triangle y}{\triangle x}$ will tend to a certain limit. In other words, $\frac{f(x + \triangle x) - f(x)}{\triangle x}$ will tend to a certain limit as $\triangle x \longrightarrow 0$. Since in the limit secant PQ, will become a tangent at P, we can call the limit of $\frac{f(x + \triangle x) - f(x)}{\triangle x}$ as $\triangle x \longrightarrow 0$, as the slope or gradient of the tangent at P. It is on account of the importance of the slope of the tangent at any point of the curve, $\underset{\triangle x \to 0}{Lt} \frac{f(x + \triangle x) - f(x)}{\triangle x}$ has a special role to play. We speak of it as on derivative of $f(x)$ with respect to x, i.e. denoted by $\frac{dy}{dx}$.

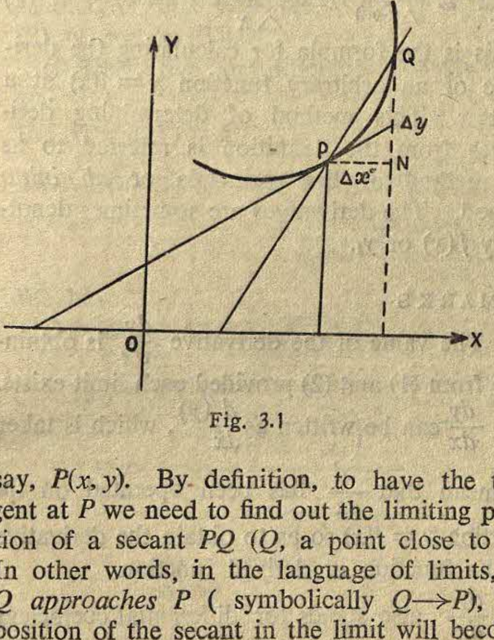


Fig. 3.1

REMARKS

1. Likewise, we can see how the idea of derivatives is necessary for study of physical quantities. For example, in order to determine the velocity of a train, we note the point at which it is located at time $t = t_1$, and then at time $t = t_2$. Let these be the distances $s = s_1$ and $s = s_2$. The increment in distance $\triangle s = s_2 - s_1$ is divided by the increment in the time $\triangle t = t_2 - t_1$. The quotient $\frac{\triangle s}{\triangle t}$ is the average velocity of the train for the interval (t_1, t_2).

But the smaller the $\triangle t$, the more exact is this speed. For this reason, the speed at time $t = t_1$ is the limit to which the ratio $\frac{\triangle s}{\triangle t}$ tends as $\triangle t \to 0$:

$$v = \underset{\triangle t \to 0}{Lt} \frac{\triangle s}{\triangle t}.$$

We can denote it as before by $\frac{ds}{dt}$.

2. Let us consider another example.

For the free fall of a body, we know,

$$s = \tfrac{1}{2} gt^2$$

Since $t_2 = t_1 + \triangle t$, it follows that

$$\triangle s = s_2 - s_1 = \tfrac{1}{2} g (t_1 + \triangle t)^2 - \tfrac{1}{2} gt_1^2$$

Hence

$$v = \underset{\triangle t \to 0}{Lt} \frac{\frac{1}{2} g(t_1 + \triangle t)^2 - \frac{1}{2} gt^2}{\triangle t} = gt_1$$

The notation t_1, is introduced so as to bring out the constancy of t when computing the limit. Since t_1 is an arbitrary value of time, the subscript 1 can best be dropped; then from the formula $v = gt$ it is evident that the velocity v (like the distance s) is a function of time. The form of the function v depends completely on the form of the function s, so that the function s generates (derives) the function v. Hence the name, the **derived function.**

3. If in $y = f(x)$, the independent variable x takes an increment $\triangle x$, then $\triangle y$ denotes the corresponding increment of y and we have

$y + \triangle y = f(x + \triangle x)$, i.e.

$\triangle y = f(x + \triangle x) - f(x)$

Now, if $y = x^2$ and x increases from 2 to 2.1, i.e. $\triangle x = 0.1$, then y increases from 4 to 4.41, i.e. $\triangle y = 0.41$. Again, when x decreases from 2 to 1.9, i.e. $\triangle x = -0.1$, then y decreases from 4 to 3.61, i.e. $\triangle y = -0.39$.

Thus for a fixed initial value 2 of x, if x increases successively to 2.1, 2.01, 2.001, etc., then although the corresponding increment $\triangle x (= .1, .01, .001 \ldots\ldots)$ and $\triangle y = (.41, .0401, .004001, \ldots\ldots\ldots)$ are getting smaller, their ratio $\frac{\triangle y}{\triangle x}$ being 4.1, 4.01, 4.001,...... is approaching a definite number 4. This illustrates the fact that the ratio can be brought as near to 4 as we please by making $\triangle x$ approach to zero. Thus the ratio of the increments $\frac{\triangle y}{\triangle x}$ has a definite finite value 4 as $\triangle x \to 0$ and consequently $\triangle y \to 0$.

With all the above examples, we can set forth the formal definition of derivative. We shall subsequently give a formal geometrical interpretation of derivatives.

3.2 The derivative (Differential Coefficient)

The derivative $\frac{dy}{dx}$ of a function $y = f(x)$ with respect to the argument x is the limit of the ratio $\frac{\triangle y}{\triangle x}$ when $\triangle x$ approaches zero; that is

$$\frac{dy}{dx} = \underset{\triangle x \to 0}{Lt} \frac{\triangle y}{\triangle x} \tag{1}$$

If $y = f(x)$ is a continuous function of x defined in the interval (a, b) and if x be some point of this interval and for every increment $\triangle x$ of x, the function $y = f(x)$ receives an increment

$$\triangle y = f(x + \triangle x) - f(x)$$

Hence

$$\frac{d(y)}{dx} = \underset{\triangle x \to 0}{Lt} \frac{f(x + \triangle x) - f(x)}{\triangle x} \tag{2}$$

This is the formula for calculating the derivative of an arbitrary function $y = f(x)$ at a point x. This method of determining derivatives from the definition is referred to as the **method of first principle** or *ab initio* method. The derivatives are sometimes denoted by $f'(x)$ or y_1.

REMARKS

1. The value of the derivative $\frac{dy}{dx}$ is obtainable from (1) and (2) provided each limit exists.

2. $\frac{dy}{dx}$ can be written as $\frac{d}{dx}(y)$, which is taken to mean that $\frac{d}{dx}$ has been operated on the function $y = f(x)$ so as to obtain the derivative of the function or derived function.

3. $\frac{dy}{dx}$ also stands for 'differential coefficient

of $y\,(=f(x))$ with respect to x' and 'differentiate $y = f(x)$ with respect to x'.

4. The function $f'(x)$ represents differential coefficient (derivative) of $f(x)$ for any x and the same for any particular value a of x is often denoted by $f'(a)$ and is defined as

$$f'(a) = \underset{\triangle x\to 0}{Lt}\ \frac{f(a + \triangle x) - f(a)}{\triangle x}$$

5. If instead of $\triangle x$ and $\triangle y$ the increments are denoted by h and k respectively, we have

$$\frac{dy}{dx}\left(= f'(x)\right) = \underset{h\to 0}{Lt}\ \frac{k}{h}$$

$$= \underset{h\to 0}{Lt}\ \frac{f(x + h) - f(x)}{h}$$

6. It is to be noted that if the differential coefficient of $f(x)$ exists at a point, then the function $f(x)$ must be continuous at that point while the converse is not always true.

3.3 Some standard results on derivatives of functions

The following standard results on derivatives of functions are obtained by applying the method of first principle as given in formula (2) of section 3.2.

(i) Derivative or differential coefficient of x^n:

Let $y(= f(x)) = x^n$

Then from the relation $f'(x)\ \left(= \dfrac{dy}{dx}\right)$

$$= \underset{\triangle x\to 0}{Lt}\ \frac{f(x + \triangle x) - f(x)}{\triangle x}$$

We have

$$f'(x) = \underset{\triangle x\to 0}{Lt}\ \frac{(x + \triangle x)^n - x^n}{\triangle x}$$

$[\because\ f(x) = x^n,\ f(x + \triangle x) = (x + \triangle x)^n]$

$$= \underset{\triangle x\to 0}{Lt}\ \frac{x^n\left(1 + \dfrac{\triangle x}{x}\right)^n - x^n}{\triangle x}$$

$$= \underset{\triangle x\to 0}{Lt}\ x^{n-1}\ \frac{\left(1 + \dfrac{\triangle x}{x}\right)^n - 1}{\dfrac{\triangle x}{x}}$$

$$= \underset{z\to 0}{Lt}\ x^{n-1}\ \frac{(1 + z)^n - 1}{z},$$

[where $z = \dfrac{\triangle x}{x}$ and as $\triangle x \longrightarrow 0,\ z \longrightarrow 0.$]

$$= x^{n-1}\ \underset{z\to 0}{Lt}\ \frac{(1 + z)^n - 1}{\log(1 + z)} \times \frac{\log(1 + z)}{z}$$

$$= x^{n-1}\ \underset{k\to 0}{Lt}\ \frac{nk}{\log(1 + k)} \times \underset{z\to 0}{Lt}\ \frac{\log(1 + z)}{z}$$

[where $(1 + z)^n = 1 + k$, $n\log(1 + z) = \log(1 + k)$ and as $z \longrightarrow 0$, $k \longrightarrow 0$]

$$= x^{n-1}\ \frac{n}{\underset{k\to 0}{Lt}\ \dfrac{1}{k}\log(1 + k)} \times \underset{z\to 0}{Lt}\ \frac{1}{z}\log(1 + z)$$

$$= x^{n-1} \times \frac{n}{1} \times 1 = nx^{n-1}$$

[using standard results on limits].

Hence $\dfrac{dy}{dx}\left(= \dfrac{dx^n}{dx}\right) = nx^{n-1}$

for all rational values of n.

Alternately when $y(= f(x)) = x^n$ we have

$$f'(x) = \underset{\triangle x\to 0}{Lt}\ \frac{(x + \triangle x)^n - x^n}{\triangle x}$$

Let $x + \triangle x = X$ so that $\triangle x = X - x$ and if $\triangle x \longrightarrow 0$, $X \longrightarrow x$

Hence

$$f'(x) = \underset{X\to x}{Lt}\ \frac{X^n - x^n}{X - x} = nx^{n-1}$$

[using the result $\underset{x\to a}{Lt}\ \dfrac{x^n - a^n}{x - a} = n\,a^{n-1}$

$\because\ \dfrac{dx^n}{dx} = n\,x^{n-1}$ for all rational values of n.

In particular, if

$n = 0 \qquad \dfrac{d\,(x^0)}{dx} = \dfrac{d\,(1)}{dx} = 0.x^{0-1} = 0$

$n = 1 \qquad \dfrac{d\,(x)}{dx} = 1.x^{1-1} = 1$

$n = 2 \qquad \dfrac{d\,(x^2)}{dx} = 2.x^{2-1} = 2x$

$n = \frac{1}{2} \qquad \dfrac{d}{dx}(x^{\frac{1}{2}}) = \dfrac{1}{2}\ x^{\frac{1}{2}-1} = \dfrac{1}{2\sqrt{x}}$

$n = -1 \qquad \dfrac{d\,(x^{-1})}{dx} = -1.\ x^{-1-1} = -\dfrac{1}{x^2}$ etc.

(ii) Differential coefficient of e^x.

Let $f(x)=e^x$. From the definition

$$f'(x)=\underset{\Delta x\to 0}{Lt}\frac{e^{x+\Delta x}-e^x}{\triangle x}=\underset{\Delta x\to 0}{Lt}\; e^x\,\frac{e^{\Delta x}-1}{\triangle x}$$

$$=e^x\underset{\Delta x\to 0}{Lt}\frac{e^{\Delta x}-1}{\triangle x}=e^x,$$

$$\left[\because\ \underset{z\to 0}{Lt}\frac{e^z-1}{z}=1\right]$$

$$\therefore\ \frac{d}{dx}\,e^x=e^x$$

(iii) Differential coefficient of e^{mx}.

Let $f(x)=e^{mx}$. From the definition

$$f'(x)=\underset{\Delta x\to 0}{Lt}\frac{e^{m(x+\Delta x)}-e^{mx}}{\triangle x}$$

$$=e^{mx}\underset{\Delta x\to 0}{Lt}\frac{e^{m\Delta x}-1}{\triangle x}$$

$$=me^{mx}\underset{\Delta x\to 0}{Lt}\frac{e^{m\Delta x}-1}{m\triangle x}$$

$$=me^{mx}$$

$$\left[\because\ \underset{z\to 0}{Lt}\frac{e^z-1}{z}=1\right]$$

$$\therefore\ \frac{d}{dx}\,e^{mx}=me^{mx}$$

In particular,

when $m=1,\ \frac{d}{dx}e^x=1.\ e^x=e^x$

$m=2,\ \frac{d}{dx}e^{2x}=2.\ e^{2x}=2e^{2x}$ etc.

(iv) Differential coefficient of $\log x$.

Let $f(x)=\log x$. Then from definition

$$f'(x)=\underset{\Delta x\to 0}{Lt}\frac{\log(x+\triangle x)-\log x}{\triangle x}$$

$$=\underset{\Delta x\to 0}{Lt}\frac{\log\left(1+\frac{\triangle x}{x}\right)}{\triangle x}$$

$$=\underset{\Delta x\to 0}{Lt}\frac{1}{x}\log\frac{\left(1+\frac{\triangle x}{x}\right)}{\frac{\triangle x}{x}}$$

$$=\frac{1}{x}\underset{z\to 0}{Lt}\frac{\log(1+z)}{z}$$

where $z=\frac{\triangle x}{x}$ and as $\triangle x\longrightarrow 0,\ z\longrightarrow 0$

$$=\frac{1}{x}\left[\because\ \underset{z\to 0}{Lt}\frac{1}{z}\log(1+z)=1\right]$$

$$\therefore\ \frac{d}{dx}(\log x)=\frac{1}{x}.$$

(v) Differential coefficient of $\sin x$.

Let $f(x)=\sin x$, then

$$f'(x)=\underset{\Delta x\to 0}{Lt}\frac{\sin(x+\triangle x)-\sin x}{\triangle x}$$

$$=\underset{\Delta x\to 0}{Lt}\frac{2\sin\frac{\triangle x}{2}\cos\left(x+\frac{\triangle x}{2}\right)}{\triangle x}$$

$$=\underset{\Delta x\to 0}{Lt}\frac{\sin\frac{\triangle x}{2}}{\frac{\triangle x}{2}}\times\underset{\Delta x\to 0}{Lt}\cos\left(x+\frac{\triangle x}{2}\right)$$

$$=1\times\cos x=\cos x$$

$$\therefore\ \frac{d}{dx}(\sin x)=\cos x$$

Similarly, $\frac{d}{dx}(\sin mx)=m\cos mx.$

(vi) Differential coefficient of $\cos x$.

Let $f(x)=\cos x$, then

$$f'(x)=\underset{\Delta x\to 0}{Lt}\frac{\cos(x+\triangle x)-\cos x}{\triangle x}$$

$$=\underset{\Delta x\to 0}{Lt}\frac{-2\sin\frac{\triangle x}{2}\sin\left(x+\frac{\triangle x}{2}\right)}{\triangle x}$$

$$=-\underset{\Delta x\to 0}{Lt}\frac{\sin\frac{\triangle x}{2}}{\frac{\triangle x}{2}}\times\underset{\Delta x\to 0}{Lt}\sin\left(x+\frac{\triangle x}{2}\right)$$

$$=-1\times\sin x=-\sin x$$

$$\therefore\ \frac{d}{dx}(\cos x)=-\sin x$$

Similarly,

$$\frac{d}{dx}(\cos mx)=-m\sin mx$$

REMARK

Other standard results are

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}(a^x) = a^x \log_e a$$

3.4 Geometrical interpretation of the derivative

Let the graph of $y = f(x)$ be represented by

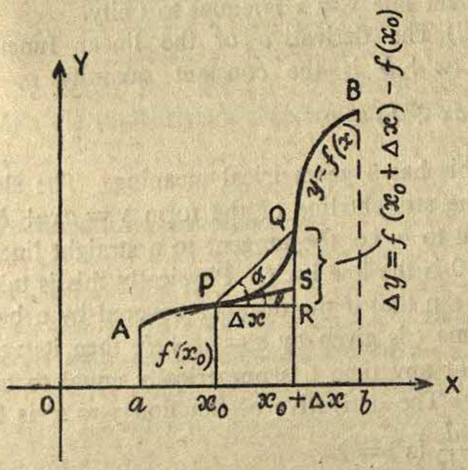


Fig. 3.2

the curve *APQB* shown in Fig. 3.2. The difference in quotient

$$\frac{\triangle y}{\triangle x} = \frac{QR}{PR} = \frac{f(x_0 + \triangle x) - f(x_0)}{\triangle x} = \tan \alpha$$

is the slope of the secant line joining points P and Q of the curve. As $\triangle x \to 0$, this secant line approaches the tangent line PS to the curve at the point P. Then

$$\underset{\triangle x \to 0}{Lt} \frac{f(x_0 + \triangle x) - f(x_0)}{\triangle x} = \underset{\triangle x \to 0}{Lt} \frac{\triangle y}{\triangle x} = \frac{SR}{PR}$$
$$= \tan \theta$$

is the slope of the tangent line to the curve at the point P. If P be any point (x, y) on the curve then we have

$$\tan \theta = \underset{\triangle x \to 0}{Lt} \frac{f(x + \triangle x) - f(x)}{\triangle x} = \frac{dy}{dx}$$
$$(= f'(x))$$

Thus, $\frac{dy}{dx}$ for any x, when it exists, gives the trigonometrical tangent of the inclination of the tangent line at the corresponding point on curve $y = f(x)$.

REMARKS

1. An equation for the tangent line to the curve at the point where $x = x_0$ is given by

$$y - f(x_0) = f'(x_0)(x - x_0)$$

2. The angle between the curves $y = f_1(x)$ and $y = f_2(x)$ at their common point (x_0, y_0) is the angle w between the tangents to the curves at the point (x_0, y_0) is obtained by using the formula of analytic geometry as

$$\tan w = \frac{f'_2(x_0) - f'_1(x_0)}{1 + f'_1(x_0)\, f'_2(x_0)}$$

3. The fact that a function can be continuous at a point and yet not be differentiable there is shown in Fig. 3.3. In this case there are two tangent lines at P represented by PM and PN. The slope of these tangent lines are $f'_-(x_0)$ and $f'_+(x_0)$ respectively. Hence no unique value of $f'(x_0)$ can be obtained at P, that is $f(x)$ is not differentiable at P.

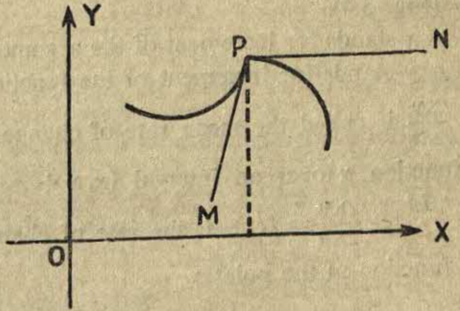


Fig. 3.3

4. If $\frac{dy}{dx}$ is positive, $\tan \theta$ is positive, i.e. θ is

acute as at P as in Fig. 3.4. At this point y increases with x.

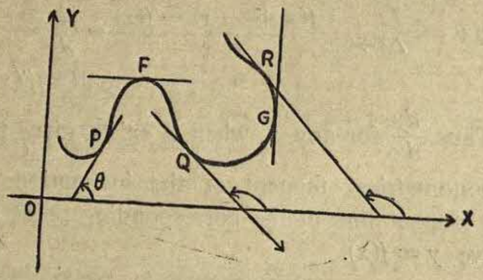


Fig. 3.4

If $\frac{dy}{dx}$, i.e. $\tan\theta$ is negative, then θ is negative as at Q or is obtained as at R (Fig. 3.4). At these points either y diminishes with increase in x or y increases with diminution in x.

If $\frac{dy}{dx}$, i.e. $\tan\theta$ is zero, then the tangent is parallel to the x-axis as at F. Again if $\frac{dy}{dx}$ $(=\tan\theta)\longrightarrow\infty$ we have $\frac{dx}{dy}\longrightarrow 0$, i.e. the tangent line at this point is parallel to y-axis as at G (Fig. 3.4).

5. $\triangle x$ stands for increment of the argument.
6. $\triangle y$ stands for increment of the function.
7. $\frac{\triangle y}{\triangle x}$ is called the mean rate of change of the function y over an interval $(x, x + \triangle x)$.
8. $\underset{\triangle x \to 0}{Lt} \frac{\triangle y}{\triangle x} = f'(x)$ is the rate of change of the function at the point x.

3.5 Derivatives of some elementary functions

(i) The derivative of a constant quantity is zero, i.e. $\frac{dc}{dx} = 0$. Physically this may be taken to mean the velocity of fixed point is zero. Geometrically, this means that the slopes of the straight line $y = a$ represented by UV in Fig. 3.5 is zero.

(ii) The derivative of an independent variable is unity, i.e. $\frac{dx}{dx} = 1$. Physically this means that if the distance covered by a body is numerically equal to the time spent in motion, then the velocity is numerically equal to unity. Geometrically this means, the slope of the straight line $y = x$ is equal to unity.

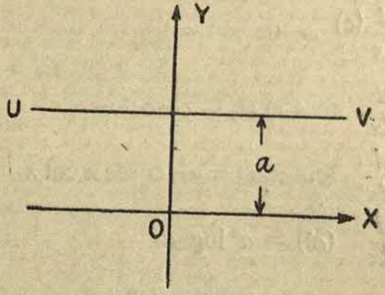


Fig. 3.5

(iii) The derivative of the linear function $y = ax + b$ is the constant quantity a, i.e. $\frac{d}{dx}(ax + b) = a$.

This has a geometrical meaning. The slope of the straight line of the form $y = ax + b$ is equal to a, i.e. the tangent to a straight line at $x = 0$ is the line itself. Physically this is taken to mean that if the distance covered by a body in time t is given by $s = at + b$, then its velocity at any time t is numerically equal to a.

(iv) The derivative of function $y = x^2$ is $2x$, i.e. $\frac{d}{dx}(x^2) = 2x$.

We have $\frac{dy}{dx}$ $(=f'(x)) = 2x$ so that at the point $(1, 1)$ $f'(x) = 2$ which represents the slope of the tangent there. Again the equation of the tangent at $(1, 1)$ is $y - f(x_0) = m(x - x_0)$, i.e. $y - 1 = 2(x - 1)$ at $(1, 1)$.

Hence, the desired slope and the equation of the tangent line at any point may be obtained from the derivative.

3.6 Fundamental theorems (or rules) for differentiation

If f, g, h are differentiable functions of x, the following differentiation rules hold :

1. $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$
$= f'(x) \pm g'(x)$.

2. $\frac{d}{dx}[cf(x)] = c \frac{d}{dx} f(x) = cf'(x)$ where c is any constant.

3. $\frac{d}{dx}[f(x) \times g(x)] = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x) = f(x) g'(x) + g(x) f'(x)$.

or, $\frac{d}{dx}$ [product of two functions] = 1st function × derivative of the second + second function × derivative of the first.

4. $\frac{d}{dx}$ { quotient of two functions }

$$= \frac{\text{(Diff. coeff. of num.)} \times \text{denom.} - \text{(Diff. coeff. of denom.)} \times \text{num.}}{\text{Square of denominator}}$$

The above four rules of operation are also called the **fundamental theorems on differentiation.**

3.7 Illustrative examples

EXAMPLE 1

If $f(x) = x^3 - \cos x + x^2 - \sin x$ then by rule 1 and standard results, we have

$$f'(x) = 3x^2 + \sin x + 2x - \cos x$$

EXAMPLE 2

If $f(x) = 5x^3 + 4x + 3 \sin x$ then by rules 1 and 2 and by standard results, we have

$$f'(x) = 10x^2 + 4 + 3 \cos x$$

EXAMPLE 3

If $y = e^x \sin x$, find $\frac{dy}{dx}$.

We have

$$\frac{dy}{dx} = \frac{d\, e^x}{dx} \sin x + e^x \frac{d}{dx} \sin x \quad \text{[by rule 3]}$$

$$= e^x \sin x + e^x \cos x$$

[by standard results]

EXAMPLE 4

If $y = \cot x$ find $\frac{dy}{dx}$.

We have

$$\frac{dy}{dx} = \frac{d}{dx} \cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$

$$= \frac{\frac{d}{dx}(\cos x) \times \sin x - \frac{d}{dx}(\sin x) \cos x}{\sin^2 x}$$

[by rule 4]

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

[by standard results]

$$= -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

$$\therefore \frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$$

EXAMPLE 5

If $y = \tan x$, find $\frac{dy}{dx}$

We have

$$\frac{dy}{dx} = \frac{d}{dx} \tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\frac{d}{dx}(\sin x) \times \cos x - \frac{d}{dx}(\cos x) \times \sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x \quad \therefore \frac{d}{dx} \tan x = \sec^2 x.$$

[NOTE: The examples above indicate how theorems or rules of differentiation could be used along with standard results to obtain derivatives of different functions.]

EXAMPLE 6

Find from first principles the derivative of a function $y = c$ where c is any constant.

Here, $f(x) = c$

$$\therefore f'(x) = \underset{\triangle x \to 0}{Lt} \frac{f(x + \triangle x) - f(x)}{\triangle x}$$

$$= \underset{\triangle x \to 0}{Lt} \frac{c - c}{\triangle x}$$

$$= \underset{\triangle x \to 0}{Lt} \frac{0}{\triangle x} = 0$$

EXAMPLE 7

Find from first principles the derivative of $\sqrt{x}$.

Here $f(x) = \sqrt{x}$

$$\therefore\ f'(x) = \underset{\triangle x\to 0}{Lt} \frac{\sqrt{x+\triangle x}-\sqrt{x}}{\triangle x}$$

$$= \underset{\triangle x\to 0}{Lt} \frac{x+\triangle x-x}{\triangle x\,(\sqrt{x+\triangle x}+\sqrt{x})}$$

(rationalising)

$$= \underset{\triangle x\to 0}{Lt} \frac{1}{\sqrt{x+\triangle x}+\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

EXAMPLE 8

Find from first principles the derivative of $x^3 + 2x$.

Here $f(x) = x^3 + 2x$

$$\therefore\ f'(x) = \underset{\triangle x\to 0}{Lt} \frac{(x+\triangle x)^3+2(x+\triangle x)-x^3-2x}{\triangle x}$$

$$= \underset{\triangle x\to 0}{Lt} \frac{\triangle x\,[3x^2+3x\triangle x+(\triangle x)^2+2}{\triangle x}$$

$$= \underset{\triangle x\to 0}{Lt} [3x^2+3x\triangle x+(\triangle x)^2+2]$$

$$= 3x^2+2$$

EXAMPLE 9

$f'(x) = \dfrac{3+x}{3-x},\ x \neq 3.$ Evaluate $f'(2)$ from definition.

$$f'(2) = \underset{\triangle x\to 0}{Lt} \frac{f(2+\triangle x)-f(2)}{\triangle x}$$

$$= \underset{\triangle x\to 0}{Lt} \frac{\left(\dfrac{5+\triangle x}{1-\triangle x}\right)-5}{\triangle x}$$

$$= \underset{\triangle x\to 0}{Lt} \frac{1}{\triangle x}\cdot\frac{6\triangle x}{1-\triangle x} = \underset{\triangle x\to 0}{Lt} \frac{6}{1-\triangle x}$$

$$= 6$$

EXAMPLE 10

If $f(x)$ has a derivative at $x = x_0$, prove that $f(x)$ must be continuous at $x = x_0$.

If we write

$$f(x_0 + \triangle x)-f(x_0) = \frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\cdot \triangle x,\ \triangle x \neq 0$$

Then

$$\underset{\triangle x\to 0}{Lt} [f(x_0+\triangle x)-f(x_0] =$$

$$\underset{\triangle x\to 0}{Lt} \frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\cdot \underset{\triangle x\to 0}{Lt} \triangle x = f'(x_0).\ 0 = 0$$

($\because$ $f'(x_0)$ exists).

Thus

$$\underset{\triangle x\to 0}{Lt} [f(x_0 + \triangle x) - f(x_0)] = 0$$

$$\text{or,}\ \underset{\triangle x\to 0}{Lt} f(x_0+\triangle x) = f(x_0)$$

showing that $f(x)$ is continuous at $x = x_0$.

[NOTE: This example shows that if a function has its derivative at a point it must be continuous at that point.]

EXAMPLE 11

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(i) Is $f(x)$ continuous at $x = 0$?

(ii) Does $f(x)$ have a derivative at $x = 0$?

(i) Here

$$\underset{x\to 0-}{Lt} f(x) = \underset{x\to 0-}{Lt} x \sin\frac{1}{x} = 0$$

$$\underset{x\to 0+}{Lt} f(x) = \underset{x\to 0+}{Lt} x \sin\frac{1}{x} = 0$$

and $f(0) = 0$ according to definition.

Hence $f(x)$ is continuous at $x = 0$.

$$\text{(ii)}\ f'(0) = \underset{\triangle x\to 0}{Lt} \frac{f(0+\triangle x)-f(0)}{\triangle x}$$

$$= \underset{\triangle x\to 0}{Lt} \frac{f(\triangle x)-f(0)}{\triangle x}$$

$$= \underset{\triangle x\to 0}{Lt} \frac{\triangle x \sin\dfrac{1}{\triangle x}-0}{\triangle x} = \underset{\triangle x\to 0}{Lt} \sin\frac{1}{\triangle x}$$

(which does not exist.)

Hence $f'(0)$ does not exist.

This example shows that a function continuous at a point may not have its derivative there.

EXAMPLE 12

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} &, x \neq 0 \\ 0 &, x = 0 \end{cases}$$

(i) Is $f(x)$ differentiable at $x = 0$?

(ii) Is $f'(x)$ continuous at $x = 0$?

(i) $f'(0) = \underset{\triangle x \to 0}{Lt} \dfrac{f(\triangle x) - f(0)}{\triangle x}$

$$= \underset{\triangle x \to 0}{Lt} \frac{(\triangle x)^2 \sin \dfrac{1}{\triangle x} - 0}{\triangle x}$$

$$= \underset{\triangle x \to 0}{Lt} (\triangle x) \sin \frac{1}{\triangle x} = 0$$

Hence $f(x)$ is differentiable at $x = 0$ and its value is 0.

(ii) From elementary differentiation rules, if $x \neq 0$.

$$f'(x) = \frac{d}{dx}\left(x^2 \sin \frac{1}{x}\right) = x^2 \frac{d}{dx}\left(\sin \frac{1}{x}\right) + \sin \frac{1}{x} \; \frac{d(x^2)}{dx}$$

$$= x^2 \cos \frac{1}{x}\left(-\frac{1}{x^2}\right) + \sin \frac{1}{x}\,(2x)$$

$$= -\cos \frac{1}{x} + 2x \sin \frac{1}{x}\ .$$

Since $\underset{x \to 0}{Lt} f'(x) = \underset{x \to 0}{Lt} \left(-\cos \dfrac{1}{x} + 2x \sin \dfrac{1}{x}\right)$ does not exist because $\underset{x \to 0}{Lt} \cos \dfrac{1}{x}$ does not exist), $f'(x)$ cannot be continuous at $x = 0$ inspite of the fact that $f'(0)$ exists.

This shows that we cannot calculate $f'(0)$ in this case by simply calculating $f'(x)$ and putting $x = 0$ as is frequently done in calculus. It is only when the derivative of a function is continuous at a point that this procedure gives the right answer. This happens to be true for most elementary functions.

EXAMPLE 13

$$f(x) = |x|$$

(i) Calculate the right hand derivative of $f(x)$ at $x = 0$.

(ii) Calculate the left hand derivative of $f(x)$ at $x = 0$.

(iii) Does $f(x)$ have a derivative at $x = 0$?

(iv) Illustrate the conclusions in (i), (ii) and (iii) by means of a graph.

(i) $f'_{+}(0) = \underset{\triangle x \to 0+}{Lt} \dfrac{f(\triangle x) - f(0)}{\triangle x} =$

$$\underset{\triangle x \to 0+}{Lt} \frac{|\triangle x| - 0}{\triangle x} = \underset{\triangle x \to 0+}{Lt} \frac{\triangle x}{\triangle x} = 1$$

since $|\triangle x| = \triangle x$ for $\triangle x > 0$.

(ii) $f'_{-}(0) = \underset{\triangle x \to 0-}{Lt} \dfrac{f(\triangle x) - f(0)}{\triangle x} =$

$$\underset{\triangle x \to 0-}{Lt} \frac{|\triangle x| - 0}{\triangle x} = \underset{\triangle x \to 0-}{Lt} \frac{-\triangle x}{\triangle x} = -1,$$

since $|\triangle x| = -\triangle x$ for $\triangle x < 0$

(iii) The derivative at $x = 0$ does not exist if the right hand and the left hand derivatives are unequal.

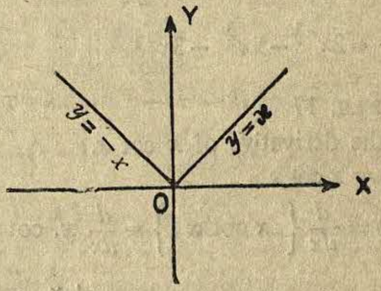


Fig. 3.6

(iv) The required graph is shown in Fig. 3.6. Note that the slopes of the lines $y = x$ and $y = -x$ and 1 are -1 respectively, representing the right and left hand derivatives at $x = 0$. However the derivative at $x = 0$ does not exist.

EXAMPLE 14

Find the derivative of $\dfrac{3x-4}{2x+3}$ at $x = 1$.

Let $f(x) = \dfrac{3x-4}{2x+3}$

Then $f'(x) = \dfrac{3(2x+3) - 2(3x-4)}{(2x+3)^2} = \dfrac{17}{(2x+3)^2}$

$\therefore\ f'(1) = \dfrac{17}{(2.1+3)^2} = \dfrac{17}{25}$

EXAMPLE 15

Find the derivative of $\sqrt[3]{x^2}$ at $x=1$.

Let $f(x) = \sqrt[3]{x^2} = x^{\frac{2}{3}}$

$$\therefore \quad f'(x) = \frac{2}{3}\, x^{\frac{2}{3}-1} = \frac{2}{3}\, x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}},$$

$$\left[\because \frac{d}{dx}\, x^n = n\, x^{n-1}\right]$$

$$\therefore \quad f'(1) = \frac{2}{3\sqrt[3]{1}} = \frac{2}{3}.$$

EXAMPLE 16

Find the derivative of $2x^{\frac{2}{3}} - 2x^{\frac{5}{2}} + x^{-3}$

Let $f(x) = 3x^{\frac{2}{3}} - 2x^{\frac{5}{2}} + x^{-3}$

$$\therefore \quad f'(x) = 3\frac{d}{dx}\, x^{\frac{2}{3}} - 2\,\frac{d}{dx}\, x^{\frac{5}{2}} + \frac{d}{dx}\, x^{-3}$$

$$= 3.\,\frac{2}{3}\, x^{\frac{2}{3}-1} - 2.\frac{5}{2}\, x^{\frac{5}{2}-1} - 3x^{-3-1}$$

$$= 2x^{-\frac{1}{3}} - 5x^{\frac{3}{2}} - 3x^{-4}$$

EXAMPLE 17

Find the derivative of x cot x.

Let $f(x) = x \cot x$

$$\therefore \quad f'(x) = \frac{d}{dx}\left\{ x \cot x \right\} = \frac{d}{dx}\, x.\, \cot x + x\,\frac{d}{dx}\cot x$$

$$= 1.\, \cot x + x\,(-\operatorname{cosec}^2 x),$$

$$\left[\because \frac{d\,x^n}{dx} = n\,x^{n-1} \text{ and } \frac{d}{dx}\cot x = -\operatorname{cosec}^2 x\right]$$

$$= \cot x - x \operatorname{cosec}^2 x$$

EXAMPLE 18

Find the derivative of $\frac{e^x}{x^2}$.

Let $f(x) = \frac{e^x}{x^2} = e^x.\; x^{-2}$

$$\therefore \quad f'(x) = \frac{d}{dx}\,(e^x)\,.\, x^{-2} + e^x \frac{d}{dx}\,(x^{-2})$$

$$= \frac{e^x}{x^2} - \frac{2\,e^x}{x^3}$$

EXAMPLE 19

Find the derivative of $(x^2-2x+2)\log x$.

Let $f(x) = (x^2-2x+2)\log x$

$$\therefore \quad f'(x) = \frac{d}{dx}\,(x^2-2x+2) \times \log x + (x^2-2x+2)\,\frac{d}{dx}\,(\log x)$$

$$= (2x-2)\log x + (x^2-2x+2) \times \frac{1}{x}$$

$$= (2x-2)\log x + \frac{x^2-2x+2}{x}$$

EXAMPLE 20

Find an equation for the tangent to $y = x^2$ at the point where (i) $x = \frac{1}{3}$, (ii) $x = 1$.

(i) $f'(x) = 2x$ so that $f'(\frac{1}{3}) = \frac{2}{3}$

Then the equation of the tangent is $f(x) - f(x_0) = f'(x_0)\,(x - x_0)$ which if $x_0 = \frac{1}{3}$ becomes

$f(x) - f(1/3) = f'(1/3)\,(x - 1/3)$
or, $f(x) - \frac{1}{9} = 2/3\,(x - 1/3)$
or, $y = 2/3\,x - 1/9$
where $y = f(x)$.

(ii) The equation of the tangent at $x = 1$ is

$f(x) - f(1) = f'(1)\,(x-1)$
i.e. $f(x) - 1 = 2(x-1)$
i.e. $y = 2x - 1$ where $y = f(x)$.

Exercise 3.1

1. Find the increment of the function $y = x^2$ that corresponds to a change in argument:
   - (i) from $x = 1$ to $x_1 = 2$
   - (ii) from $x = 1$ to $x_1 = |0|$
   - (iii) from $x = 1$ to $x_1 = 1 + h$
2. Find $\triangle y$ of the function $y = \sqrt[3]{x}$ if,
   - (i) $x = 0$, $\triangle x = 0.001$
   - (ii) $x = 8$, $\triangle x = -9$
   - (iii) $x = a$, $\triangle x = h$
3. Find the increment $\triangle y$ and the ratio $\frac{\triangle y}{\triangle x}$ for the functions
   - (i) $y = \frac{1}{(x^2-2)^2}$ for $x = 1$ and $\triangle x = 0.4$
   - (ii) $y = \sqrt{x}$ for $x = 0$ and $\triangle x = 0.0001$

4. Find the slope of the secant to the parabola $y = 2x - x^2$ if the abscissae of the points of intersection are equal:

(i) $x_1 = 1,\ x_2 = 2$
(ii) $x_1 = 1,\ x_2 = 0.9$
(iii) $x_1 = 1,\ x_2 = 1 + h$

To what limit does the slope of the secant tend in the latter case if $h \to 0$?

5. What is the mean rate of change of the function $y = x^3$ in the interval $1 \leqslant x \leqslant 4$?

6. The law of motion of a point is $s = 2t^2 + 3t + 5$, where the distance s is given in cm. and the time t in sec. What is the average velocity of the point over the interval of time from $t = 1$ to $t = 5$?

7. Find the mean rise of the curve $y = 2^x$ in the interval $[x, x + \triangle x]$.

8. What is to be understood by the rise of the curve $y = f(x)$ at a given point x?

9. A hot body placed in a medium of lower temperature cools off. What is to be understood by (i) the mean rate of cooling, (ii) the rate of cooling at a given instant?

10. Find the derivative of the function $y = \tan x$ from the definition.

11. Find $\dfrac{dy}{dx} = \underset{\triangle x \to 0}{Lt} \dfrac{\triangle y}{\triangle x}$ of the functions

(i) $y = x^3$, (ii) $y = \sqrt{x}$
(iii) $y = \dfrac{1}{x^2}$, (iv) $y = \cot x$

12. Calculate $f'(8)$, if $f(x) = \sqrt[3]{x}$.

13. Find $f'(0)$, $f'(1)$, $f'(2)$ if
$f(x) = x(x-1)^2(x-2)^3$

14. At what points does the derivative of the function $f(x) = x^3$ coincide numerically with the value of the function itself, that is, $f(x) = f'(x)$?

15. The law of motion of a point is $s = 5t^2$, where the distance s is in meters and the time t is in seconds. Find the speed at $t = 3$.

16. Find the slope of the tangent to the curve $y = 0.1x^3$ drawn at a point with the abscissa $x = 2$.

17. Find the slope of the tangent to the curve $y = \sin x$ at the point $(\pi, 0)$.

18. What are the slopes of the tangents to the curves $y = \dfrac{1}{x}$ and $y = x^2$ at the point of their intersection? Find the angle between these tangents.

19. Find the equation of the tangent to the curve $y = \sin x$ at the point $(0, 0)$.

20. $f(x) = \sqrt{2x-1}$. Evaluate $f'(5)$ from the definition.

21. Differentiate the following functions w, r, t x:

(i) $y = ax^m + bx^{m+n}$ (ii) $y = \dfrac{a}{\sqrt[3]{x^2}} - \dfrac{b}{x\sqrt[3]{x}}$
(iii) $y = \dfrac{a+bx}{c+dx}$ (iv) $y = \dfrac{2}{2x-1} - \dfrac{1}{x}$
(v) $y = \dfrac{1+\sqrt{x}}{1-\sqrt{x}}$ (vi) $y = \tan x - \cot x$
(vii) $y = \dfrac{\sin x + \cos x}{\sin x - \cos x}$
(viii) $y = 2x \sin x - (x^2-2) \cos x$
(ix) $y = (x-1)e^x$ (x) $y = \dfrac{x^2}{\log x}$
(xi) $y = \dfrac{x^5}{e^x}$ (xii) $y = x^3 \log x - \dfrac{x^3}{3}$
(xiii) $y = \dfrac{1}{x} + 2 \log x - \dfrac{\log x}{x}$
(xiv) $y = e^x \cos x$

22. $f(x) = \begin{cases} x^3 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Prove that (i) $f(x)$ is continuous at $x = 0$, (ii) $f(x)$ has a derivative at $x = 0$, (iii) $f'(x)$ is continuous at $x = 0$.

23. $f(x) = \begin{cases} xe^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Determine whether $f(x)$ (i) is continuous at $x = 0$, (ii) has a derivative at $x = 0$.

24. If $f(x) = e^x$, show that $f'(a) = e^a$ depends on the result $\underset{\triangle x \to 0}{Lt} \dfrac{e^{\triangle x} - 1}{\triangle x} = 1$.

25. Use the results $\underset{\triangle x\to 0}{Lt}\ \frac{\sin\triangle x}{\triangle x}=1$, $\underset{\triangle x\to 0}{Lt}\ \frac{1-\cos\triangle x}{\triangle x}=0$ to prove that if $f(x)=\sin x$, $f'(a)=\cos a$.

26. $f(x)=x\,|\,x\,|$,

(i) Calculate the right hand derivative of $f(x)$ at $x=0$.

(ii) Calculate the left hand derivative of $f(x)$ at $x=0$.

(iii) Does $f(x)$ have a derivative at $x=0$?

(iv) Illustrate the conclusions in (i), (ii) and (iii) from a graph.

27. $f(x)=\begin{cases} 2x-3, & 0\leqslant x\leqslant 2 \\ x^2-3, & 2<x\leqslant 4. \end{cases}$

Discuss (i) the continuity and (ii) the differentiability of $f(x)$ in $0\leqslant x\leqslant 4$.

28. At what point is the tangent to the parabola $y=x^2-7x+3$ parallel to the straight line $5x+y-3=0$?

29. Determine the slope of the tangent to the curve $x^3+y^3-xy-7=0$ at the point (1, 2).

30. Write the equation of the tangent and the normal to the parabola $y=\sqrt{x}$ at the point with abscissa $x=4$.

3.8 Derivative of the function of a function (Composite Function)

A quantity y is called a **function of a function** (composite function) if it is regarded as the function of some (auxiliary) variable u, which, in turn, is a function of an argument x. In symbols, we have, if $y=f(u)$ and $u=\phi(x)$ then y is a function of x and this may be written as $y=f[\phi(x)]$.

REMARK

If $f(u)$ and $\phi(x)$ are continuous functions, then the function $f[\phi(x)]$ is also continuous.

The **derivative of a function of a function** is equal to the derivative of the function with respect to the auxiliary variable multiplied by the derivative of the auxiliary variable with respect to the argument. Symbolically we write

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} \qquad (1)$$

For example to find the derivative of the function

$$y=\sqrt{a^2-x^2}$$

with respect to x, we write it as

$$y=u^{\frac{1}{2}} \quad \text{and } u=a^2-x^2$$

so that

$$\frac{dy}{du}=\frac{1}{2}\,u^{\frac{1}{2}-1}=\frac{1}{2\sqrt{u}}$$

$$\text{and} \quad \frac{du}{dx}=-2x$$

$$\text{Hence } \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{2\sqrt{u}}\times(-2x)$$

$$=\frac{-x}{\sqrt{a^2-x^2}}$$

As another example, to find the derivative of the function $y=\sin^2 2x$ we may write this function as a chain of three relations:

$$y=u^2,\ u=\sin v,\ v=2x$$

This gives

$$\frac{dy}{du}=2u,\quad \frac{du}{dv}=\cos v,\quad \frac{dv}{dx}=2$$

Now by extending the definition in (1) we have

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dv}\times\frac{dv}{dx}$$

Hence in this case

$$\frac{dy}{dx}=2u\times\cos v\times 2=4u\cos v=$$

$$4\sin 2x\cos 2x=2\sin 4x$$

3.9 Derivative of the inverse function

If from the relation $y=f(x)$ there follows the relation $x=\phi(y)$, then the function $\phi(y)$ is called an inverse of $f(x)$.

For example the inverse of the function $y=\sin x$ is the function $x=\sin^{-1}y$.

The *derivative* of an inverse function is equal to unity divided by the derivative of the original function.

$$\frac{dx}{dy} = \frac{1}{dy/dx} \text{ or, } \frac{dy}{dx} = \frac{1}{dx/dy}. \qquad (2)$$

For example, to find the derivative of the function $y = \sin^{-1} x$ we may write it as $x = \sin y$, so that $\frac{dx}{dy} = \cos y$ i.e.

$$\frac{dx}{dy} = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

Hence from (2) $\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\sqrt{1 - x^2}}$

The result is valid for $-1 < x < 1$.

3.10 The derivative of an implicit function

If a relationship between x and y is given in the form $F(x, y) = 0$ — (3), then the function is said to have an **implicit form** and under certain conditions, we can write $y = f(x)$.

To find the derivative $\frac{dy}{dx}$ of the function $x^3 + y^3 - 3axy = 0$ we have, on equating the derivative of the left hand side of the equation to zero,

$$3x^2 + 3y^2 \frac{dy}{dx} - 3a\left(y + x\frac{dy}{dx}\right) = 0$$

This on solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = \frac{x^2 - ay}{ax - y^2}$$

3.11 Derivatives of functions represented parametrically

If a function y is related to an argument x by means of a parameter t,

$$\begin{cases} x = \phi(t) \\ y = \psi(t) \end{cases}$$

Then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

To find the derivative $\frac{dy}{dx}$ of the function represented as $x = a \cos t$, $y = a \sin t$ we have

$$\frac{dx}{dt} = -a \sin t, \quad \frac{dy}{dt} = a \cos t$$

so that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \cos t}{-a \sin t} = -\cot t$$

3.12 Another method of obtaining derivatives

To find the derivative of the function $y = u^v$ where $u = \phi(x)$ and $v = \psi(x)$, we have, on taking logarithms

$$\log y = v \log u$$

Differentiating both sides of this equation with respect to x

$$\frac{d}{dx}(\log y) = \frac{dv}{dx} \log u + v \frac{d}{dx} \log u$$

or, $\frac{1}{y}\frac{dy}{dx} = v_1 \log u + \frac{v}{u} u_1$

$$\therefore \quad y_1 = y\left(v_1 \log u + \frac{v}{u} u_1\right)$$

$$\left(\text{where } \frac{dv}{dx} = v_1, \text{ etc.}\right)$$

$$= u^v\left(v_1 \log u + \frac{v}{u} u_1\right).$$

As an example, to obtain the derivative of $y = (\sin x)^x$ we have on taking logarithms

$$\log y = x \log \sin x$$

Thus on differentiating w.r.t. x on either side, we get

$$\frac{1}{y}\frac{dy}{dx} = \log \sin x + x \cot x$$

$$\therefore \quad \frac{dy}{dx} = (\sin x)^x [\log \sin x + x \cot x].$$

3.13 Derivatives of higher orders

A derivative of the second order, or the second derivative of the function $y = f(x)$ is the derivative of its derivative, that is

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

The second derivative may be denoted as

$$y_2 \text{ or } \frac{d^2y}{dx^2} \text{ or } f''(x)$$

If $x = f(t)$ is the law of rectilinear motion of a point, then $\frac{d^2x}{dt^2}$ is the acceleration of this motion. Generally, the nth derivative of a function $y = f(x)$ is the derivative of a derivative of order $(n - 1)$. For the nth derivative we use the notation

$$y_n \text{ or } \frac{d^ny}{dx^n} \text{ or } f^n(x)$$

For example to find the second derivative of the function $y = \log(1 - x)$ we have

$$\frac{dy}{dx} = \frac{-1}{1-x}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left\{\frac{-1}{1-x}\right\}$$

$$= \frac{0\times(1-x)-(-1)\times(-1)}{(1-x)^2} = \frac{-1}{(1-x)^2}$$

3.13 (*a*) *Leibniz rule*

If the function $u = \phi(x)$ and $v = \psi(x)$ have derivatives up to the nth order inclusive, then to evaluate the nth derivative of a product of these functions we can use the Leibniz rule (or formula):

$$(uv)_n = u_n v + n u_{n-1} v_1 + \frac{n\,(n-1)}{2\,!}\,u_{n-2}v_2 + \ldots\ldots + uv_n$$

3.13 (*b*) *Higher-order derivatives of functions represented parametrically*

If $\begin{cases} x = \phi(t) \\ y = \psi(t) \end{cases}$

Then the derivatives $y_1 = \dfrac{dy}{dx}$, $y_2 = \dfrac{d^2y}{dx^2}$ can successively be calculated by the formulas

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt}$$

$$\frac{d^3y}{dx^3} = \frac{\dfrac{d}{dt}\left(\dfrac{d^2y}{dx^2}\right)}{\dfrac{dx}{dt}}$$

and so forth.

For a second derivative we have the formula

$$\frac{d^2y}{dx^2} = \frac{\dfrac{dx}{dt}\,\dfrac{d^2y}{dt^2} - \dfrac{d^2x}{dt^2}\,\dfrac{dy}{dt}}{\left(\dfrac{dx}{dt}\right)^3}$$

i.e. $y_2 = \dfrac{x_t y_{tt} - x_{tt} y_t}{(x_t)^3}$

For example if $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$

We have

$$y_1 = \frac{y_t}{x_t} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot t$$

$$\text{and } y_2 = \frac{(y_1)_t}{x_t} = \frac{-\dfrac{b}{a}\cdot\dfrac{-1}{\sin^2 t}}{-a\sin t} = -\frac{b}{a^2\sin^3 t}$$

3.13 (*c*) *Interpretation of the second derivative*

Let a point be in rectilinear motion. Traversing a distance s in time t, it acquires a velocity v. Let this velocity change, the increment during the time interval $(t, t + \triangle t)$ being $\triangle v$. Then the ratio $\dfrac{\triangle v}{\triangle t}$ yields the change in velocity per (average) unit of time and is called the average acceleration. This relation describes the rate of change of the velocity at time t. Therefore, the acceleration (at time t) is the limit of the ratio $\dfrac{\triangle v}{\triangle t}$ as $\triangle t \to 0$, that is, the derivative $\dfrac{dv}{dt}$. Therefore, acceleration is the (first) derivative with respect to the time of the velocity which in turn is the (first) derivative of distance with respect to time and hence, acceleration is the second derivative of the distance with respect to the time.

3.14 Illustrative examples

EXAMPLE 1

Find the derivative of $y = (1 + 3x - 5x^2)^{30}$.

Let $y = u^{30}$ and $u = 1 + 3x - 5x^2$

$$\therefore \frac{dy}{du} = 30u^{29} \text{ and } \frac{du}{dx} = 3 - 10x$$

Hence

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 30u^{29}(3 - 10x)$$

$$= 30(3 - 10x)(1 + 3x - 5x^2)^{29}$$

EXAMPLE 2

Differentiate $y = \tan^3 x$ with respect to x.

Here let $y = u^3$ and $u = \tan x$

$$\therefore \frac{dy}{du} = 3u^2 \text{ and } \frac{du}{dx} = \sec^2 x$$

Hence

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 3u^2 \sec^2 x = 3 \tan^2 x \sec^2 x.$$

EXAMPLE 3

Differentiate $y = \sin \sqrt{x}$ with respect to x.

Let $y = \sin u$ and $u = \sqrt{x}$

$$\therefore \quad \frac{dy}{du} = \cos u \quad \text{and} \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

Hence

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \times \frac{1}{2\sqrt{x}}$$

$$= \frac{\cos \sqrt{x}}{2\sqrt{x}}$$

EXAMPLE 4

Find the derivative of $y = \log \sin (ax + b)$.

Let $y = \log u$, $u = \sin v$ and $v = ax + b$

$$\therefore \quad \frac{dy}{du} = \frac{1}{u}, \quad \frac{du}{dv} = \cos v \text{ and } \frac{dv}{dx} = a$$

Hence

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx} = \frac{1}{u} . \cos v. \, a$$

$$= \frac{a \cos (ax+b)}{\sin (ax+b)}$$

EXAMPLE 5

Find the derivative of $y = e^{\sin^2 x}$.

Let $y = e^u$, $u = v^2$ and $v = \sin x$

$$\therefore \quad \frac{dy}{dx} = e^u, \quad \frac{du}{dv} = 2v \text{ and } \frac{dv}{dx} = \cos x$$

Hence

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx} = e^u \times 2v \times \cos x$$

$$= 2\, e^{\sin^2 x} \sin x \cos x = e^{\sin^2 x} \sin 2x.$$

EXAMPLE 6

Find the derivative of $y = \cos^{-1} x$.

Here $x = \cos y$

$$\therefore \quad \frac{dx}{dy} = -\sin y = -\sqrt{1 - \cos^2 y} = -\sqrt{1 - x^2}$$

$$\therefore \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}, \; -1 < x < 1$$

EXAMPLE 7

Find the derivative of $y = \tan^{-1} x$.

Here $x = \tan y$

$$\therefore \quad \frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1+x^2}$$

REMARKS

1. $\dfrac{d}{dx} \sin^{-1} x = \dfrac{1}{\sqrt{1 - x^2}}, \quad -1 < x < 1 \text{ or } |x| < 1$
2. $\dfrac{d}{dx} \cos^{-1} x = \dfrac{-1}{\sqrt{1 - x^2}}, \quad -1 < x < 1 \text{ or } |x| < 1$
3. $\dfrac{d}{dx} \tan^{-1} x = \dfrac{1}{1 + x^2}$
4. $\dfrac{d}{dx} \cot^{-1} x = \dfrac{-1}{1 + x^2}$
5. $\dfrac{d}{dx} \sec^{-1} x = \dfrac{1}{x\sqrt{x^2 - 1}}, \quad |x| > 1$
6. $\dfrac{d}{dx} \operatorname{cosec}^{-1} x = \dfrac{-1}{x\sqrt{x^2 - 1}}, \quad |x| > 1$

EXAMPLE 8

Find the derivative $\dfrac{dy}{dx}$ if $x^3 + y^3 - 3axy = 0$.

Forming the derivative of the left hand side and equating it to zero we get

$$3x^2 + 3y^2 \frac{dy}{dx} - 3a\left(y + x\frac{dy}{dx}\right) = 0$$

$$\text{or,} \quad \frac{dy}{dx} = \frac{x^2 - ay}{ax - y^2}$$

EXAMPLE 9

Calculate $\dfrac{dy}{dx}$ if $\tan y = xy$.

Differentiating either side with respect to x we have, $\sec^2 y \dfrac{dy}{dx} = y + x \dfrac{dy}{dx}$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{\sec^2 y - x} = \frac{y}{1 + \tan^2 y - x}$$

$$= \frac{y}{1 + x^2y^2 - x}$$

EXAMPLE 10

Find $\frac{dy}{dx}$, if $x = a(\cos t + t\sin t)$

$$y = a(\sin t - t\cos t).$$

Here

$$\frac{dx}{dt},= a[-\sin t + \sin t + t\cos t] = a\,t\cos t$$

and $\frac{dy}{dt} = a[\cos t - \cos t + t\sin t] = a\,t\sin t$

$$\therefore\ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\,t\sin t}{a\,t\cos t} = \tan t$$

EXAMPLE 11

Find $\frac{dy}{dx}$ when $t = \frac{\pi}{4}$ if $\begin{cases} x = e^t\cos t \\ y = e^t\sin t \end{cases}$

Here $\frac{dy}{dt} = e^t\sin t + e^t\cos t = e^t(\sin t + \cos t)$

$$\frac{dx}{dt} = e^t\cos t - e^t\sin t = e^t(\cos t - \sin t$$

$$\therefore\ \frac{dy}{dx} = \frac{e^t(\sin t + \cos t)}{e^t(\cos t - \sin t)} = \frac{\sin t + \cos t}{\cos t - \sin t}$$

$\therefore \left[\frac{dy}{dx}\right]_{t = \frac{\pi}{4}} = \frac{\sqrt{2}}{0}$ = a meaningless quantity

and $\underset{t\to\frac{\pi}{4}}{Lt}\frac{dy}{dx} = \underset{t\to\frac{\pi}{4}}{Lt}\frac{\sin t + \cos t}{\cos t - \sin t} = \infty$

EXAMPLE 12

Calculate $\frac{dy}{dx}$ when $t = \frac{\pi}{2}$, if

$$x = a(t - \sin t)$$
$$y = a(1 - \cos t)$$

Here

$$\frac{dy}{dt} = a\sin t,\ \frac{dx}{dt} = a(1 - \cos t)$$

$$\therefore\ \frac{dy}{dx} = \frac{a\sin t}{a(1 - \cos t)} = \frac{\sin t}{1 - \cos t}$$

$$\therefore\ \left[\frac{dy}{dx}\right]_{t = \frac{\pi}{2}} = \frac{\sin\frac{\pi}{2}}{1 - \cos\frac{\pi}{2}} = \frac{1}{1} = 1$$

EXAMPLE 13

Find $\frac{dy}{dx}$ of $y = (\sin x)^x$.

Here $\log y = x\log\sin x$

Differentiating either side w.r.t. x we have

$$\frac{1}{y}\frac{dy}{dx} = \log\sin x + x\frac{\cos x}{\sin x}$$
$$= \log\sin x + x\cot x$$
$$\therefore\ \frac{dy}{dx} = y\left\{\log\sin x + x\cot x\right\}$$
$$= (\sin x)^x\{\log\sin x + x\cot x\}$$

EXAMPLE 14

Find the second derivative $\left(\frac{d^2y}{dx^2}\right)$ of the function $y = x^8 + 7x^6 - 5x + 4$.

Here $\frac{dy}{dx} = 8x^7 + 7.6x^5 - 5 =$

$$8x^7 + 42x^5 - 5$$

$$\therefore\ \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 8.7x^6 + 42.5x^4 =$$

$$56x^6 + 210x^4.$$

EXAMPLE 15

Find the second derivative of $y = \sin^2 x$.

Let $y = u^2$ and $u = \sin x$

So that $\frac{dy}{du} = 2u$ and $\frac{du}{dx} = \cos x$

$$\therefore\ \frac{dy}{dx} = 2u,\ \cos x = 2\sin x\cos x$$

Here

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left\{2\sin x\cos x\right\}$$
$$= 2\frac{d}{dx}\left\{\sin x\cos x\right\}$$
$$= 2[\cos x,\ \cos x + \sin x(-\sin x)]$$
$$= 2[\cos^2 x - \sin^2 x] = 2\cos 2x$$

EXAMPLE 16

Find $\frac{d^2y}{dx^2}$ of the function $x = \tan^{-1} t$

$$y = \log(1 + t^2)$$

Here

$$\frac{dx}{dt} = \frac{1}{1+t^2} \text{ and } \frac{dy}{dt} = \frac{2t}{1+t^2},$$

[For $\frac{dy}{dt}$, we represent y as $y = \log u$ and $u = 1 + t^2$]

$$\therefore \quad \frac{dy}{dx} = \frac{\dfrac{2t}{1+t^2}}{\dfrac{1}{1+t^2}} = 2t$$

Hence

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \Big/ \frac{dx}{dt}$$

$$= \frac{2}{\dfrac{1}{1+t^2}} = 2(1+t^2).$$

EXAMPLE 17

Find y_2 if $x^2 + y^2 = 1$.

By the rule for differentiation of a composite function we have $2x + 2y\ y_1 = 0$

i.e., $y_1 = -\frac{x}{y}$ where $y_1 \equiv \frac{d}{dx}$

$$\therefore \quad y_2 = -\ \frac{1.y - y_1.x}{y_2} = -\ \frac{y - xy_1}{y^2}$$

where $y_2 \equiv \frac{d^2}{dx^2}$

We finally get

$$y_2 = \frac{-\left\{ y - x\left(-\frac{x}{y}\right) \right\}}{y^2}$$

$$= \frac{-\left\{ \dfrac{y^2+x^2}{y} \right\}}{y^2} = -\ \frac{x^2+y^2}{y^3} = -\frac{1}{y^3}$$

EXAMPLE 18

Find the nth derivative of the function $y = (ax + b)^n$.

Here

$y = (ax + b)^n$

$y_1 = n(ax + b)^{n-1}.\ a$

(by the rule of composite function)

$y_2 = n(n - 1)(ax + b)^{n-2}.\ a^2$

$y_3 = n(n - 1)(n - 2)(ax + b)^{n-3}.\ a^3$

....

....

$y_n = n(n - 1)(n - 2)......[n - (n - 1)](ax + b)^{n-n}a^n$

$= \{ n(n - 1)(n - 2).........1 \}.\ a^n$

$= a^n \lfloor n$

REMARKS

1. Putting $a = 1$ and $b = 0$ we have

$$\frac{d^n}{dx^n}\left(x^n \right) = \lfloor n$$

2. $$\frac{d^n}{dx^n}(ax+b)^m = \frac{\lfloor m}{\lfloor m-n}\ a^n\ (ax+b)^{m-n}, \quad m > n$$

$$= a^n \lfloor n \quad , m=n$$

$$= 0 \quad , m<n$$

EXAMPLE 19

Find the nth derivative of $y = \sin x$.

Here $y = \sin x$

$y_1 = \cos x = \sin\left\{\frac{\pi}{2} + x\right\}$

$y_2 = -\sin x = \sin\left\{\pi + x\right\}$

$= \sin\left\{2\frac{\pi}{2} + x\right\}$

$y_3 = -\cos x = \sin\left\{3\frac{\pi}{2} + x\right\}$

....

....

$y_n = \sin\left\{n.\ \frac{\pi}{2} + x\right\}$

$$\therefore \quad \frac{d^n}{dx^n}\left(\sin x\right) = \sin\left\{n.\ \frac{\pi}{2} + x\right\}$$

Similarly,

$$\frac{d^n}{dx^n}\left(\cos x\right) = \cos\left\{n.\ \frac{\pi}{2} + x\right\}$$

REMARKS

1. $$\frac{d^n}{dx^n}\left\{\sin(ax+b)\right\} = a^n \sin\left\{\frac{n\pi}{2} + (ax+b)\right\}$$

2. $$\frac{d^n}{dx^n}\left\{\cos(ax+b)\right\} = a^n \cos\left\{\frac{n\pi}{2} + (ax+b)\right\}$$

EXAMPLE 20

Few more standard results on *n*th derivatives.

(i) $\frac{d^n (e^{ax})}{dx^n} = a^n e^{ax}$

(ii) $\frac{d^n}{dx^n}\left(\frac{1}{x+a}\right) = \frac{(-1)^n \lfloor n}{(x+a)^{n+1}}$

(iii) $\frac{d^n}{dx^n}\left\{\log (x+a)\right\} = \frac{(-1)^{n-1} \lfloor n-1}{(x+a)^n}$

EXAMPLE 21

Find the *n*th derivative of the function $y = e^{ax}x^3$.

Here, let, $u = e^{ax}$, $v = x^3$

so that $u_n = a^n e^{ax}$ and $v_1 = 3x^2$, $v_2 = 6x$,

$v_3 = 6$, $v_4 = v_5 = \ldots\ldots = 0$

By Leibniz rule, we have

$$y_n = a^n e^{ax} + n.\, a^{n-1}e^{ax}.\, 3x^2 + \frac{n(n-1)}{2!} a^{n-2}e^{ax}.\, 6x + \frac{n(n-1)(n-2)}{3!} a^{n-3} e^{ax}.\, 6$$

$$= e^{ax}a^{n-3}\left\{ a^3x^3 + 3na^2x^2 + 3n(n-1)\, ax + n(n-1)(n-2)\right\}$$

EXAMPLE 22

Differentiate *n* times the expression

$$(1 - x^2)\, y_2 - xy_1 - 2 = 0$$

By Leibniz rule we have

$$\frac{d^n}{dx^n}\left\{(1-x^2)\, y_2\right\} = y_{n+2}(1 - x^2) + n.y_{n+1}(-2x) + \frac{n(n-1)}{2!} y_n(-2)$$

$$\frac{d^n}{dx^n}\left\{xy_1\right\} = y_{n+1}.x + n.y_n.1$$

Hence the given expression after *n*th times differentiation gives

$$(1 - x^2)\, y_{n+2} - (1 + 2n)\, xy_{n+1} - \left\{n(n-1) + n\right\} y_n = 0$$

i.e. $(1 - x^2)\, y_{n+2} - (1 + 2n)\, xy_{n+1} - n^2y_n = 0$

Exercise 3.2

1. Find the derivatives of the following functions:

(i) $y = \left(\frac{ax+b}{c}\right)^3$ (ii) $y = (3 + 2x^2)^4$

(iii) $y = \sqrt{1 - x^2}$ (iv) $y = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}}$

(v) $y = (3 - 2 \sin x)^5$

(vi) $y = \tan x - \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x$

(vii) $y = \sqrt{\cot x} - \sqrt{\cot \alpha}$

(viii) $y = \operatorname{cosec}^2 x + \sec^2 x$

(ix) $y = \sqrt[3]{\sin^2 x} + \frac{1}{\cos^3 x}$

(x) $y = \sqrt{xe^x + x}$

(xi) $y = \sqrt[3]{2e^x + 2^x + 1}$

(xii) $y = \sin 3x + \cos \frac{x}{5} + \tan \sqrt{x}$

(xiii) $y = \sin (x^2 - 5x + 1) + \tan \frac{x}{a}$

(xiv) $y = \sin^{-1} \frac{1}{x^2}$ (xv) $y = \cos^{-1} \sqrt{x}$

(xvi) $y = x^2\, 10^{2x}$ (xvii) $y = \log \sin x$

(xviii) $y = \log (1 - x^2)$ (xix) $f(t) = t \sin 2^t$

(xx) $y = \tan^{-1} (\log x) + \log\, (\tan^{-1} x)$

(xxi) $y = \log^2 x - \log (\log x)$

(xxii) $y = \sqrt{\log x + 1} + \log\, (\sqrt{x} + 1)$

(xxiii) $y = \sqrt{e^{ax}}$

(xxiv) $y = e^{\sin^2 x}$ (xxv) $f(t) = e^{\alpha t} \cos \beta\, t$

(xxvi) $y = \log \frac{(x-2)^5}{(x+1)^3}$

(xxvii) $f(x) = \frac{x \sin^{-1} x}{\sqrt{1-x^2}} + \log\, \sqrt{1-x^2}$

(xxviii) $y = \log \frac{1 + \sqrt{\sin x}}{1 - \sqrt{\sin x}} + 2 \tan^{-1} \sqrt{\sin x}$

2. Find $y_1 \left(= \frac{dy}{dx}\right)$, if

(i) $y = |x|$

(ii) $y = x|x|$

Construct the graphs of the functions y and y_1.

3. Find $\frac{dy}{dx}$ if $y = \log|x|$, $(x \neq 0)$.

4. Find $f'(x)$ if $f(x) = \begin{cases} 1 - x & \text{for } x \leqslant 0 \\ e^{-x} & \text{for } x > 0. \end{cases}$

5. Calculate $f'(0)$ if $f(x) = e^{-x} \cos 3x$.

6. $f(x) = \log (1 + x) + \sin^{-1} \frac{x}{2}$; find $f'(1)$.

7. $y = \tan^3 \frac{\pi x}{6}$; find $\left(\frac{dy}{dx}\right)_{x=2}$.

8. Find $f(0) + xf'(0)$ of the function $f(x) = e^{-x}$.

9. Given the functions $f(x) = \tan x$ and $\varphi(x) = \log(1 - x)$ find $f'(0)/\varphi'(0)$.

10. Show that the function $y = xe^{-x}$ satisfies the equation $xy_1 = (1 - x)y$.

11. Show that the function $y = xe^{-\frac{x^2}{2}}$ satisfies the equation $xy_1 = (1 - x^2)y$.

12. In the following problems find $\frac{dy}{dx}$ after first taking logs of the function $y = f(x)$:

(i) $y = (x + 1)(2x + 1)(3x + 1)$

(ii) $y = \sqrt[x]{x}$

(iii) $y = \dfrac{(x + 2)^2}{(x + 1)^3 (x + 3)^4}$ (iv) $y = x^{\sqrt{x}}$

(v) $y = \sqrt{\dfrac{x(x - 1)}{x - 2}}$ (vi) $y = x^{x^x}$

(vii) $y = x \sqrt[3]{\dfrac{x^2}{x^2+1}}$ (viii) $y = x^{\sin x}$

(ix) $y = (\cos x)^{\sin x}$ (x) $y = \left(1 + \dfrac{1}{x}\right)^x$

(xi) $y = x^x$ (xii) $y = x^x$

(xiii) $y = (\tan^{-1} x)^x$

13. Find $y_1 = \frac{dy}{dx}$ of the following functions represented parametrically:

(i) $\begin{cases} x = 2t - 1 \\ y = t^3 \end{cases}$ (ii) $\begin{cases} x = \cos^2 t \\ y = b \sin^2 t \end{cases}$

(iii) $\begin{cases} x = \dfrac{1}{t+1} \\ y = \left(\dfrac{t}{t+1}\right)^2 \end{cases}$ (iv) $\begin{cases} x = a \cos^3 t \\ y = b \sin^3 t \end{cases}$

(v) $\begin{cases} x = \dfrac{2at}{1+t^2} \\ y = \dfrac{a(1-t^2)}{1+t^2} \end{cases}$ (vi) $\begin{cases} x = \dfrac{\cos^3 t}{\sqrt{\cos 2t}} \\ y = \dfrac{\sin^3 t}{\sqrt{\cos 2t}} \end{cases}$

(vii) $\begin{cases} x = \dfrac{3at}{1+t^3} \\ y = \dfrac{3at^2}{1+t^3} \end{cases}$ (viii) $\begin{cases} x = \cos^{-1} \dfrac{1}{\sqrt{1+t^2}} \\ y = \sin^{-1} \dfrac{t}{\sqrt{1+t^2}} \end{cases}$

(ix) $\begin{cases} x = \sqrt{t} \\ y = \sqrt[3]{t} \end{cases}$ (x) $\begin{cases} x = e^{-t} \\ y = e^{2t} \end{cases}$

(xi) $\begin{cases} x = \sqrt{t^2 + 1} \\ y = \dfrac{t-1}{\sqrt{t^2+1}} \end{cases}$

(xii) $\begin{cases} x = a(\cos t + t \sin t) \\ y = a(\sin t - t \cos t) \end{cases}$

14. Calculate $\frac{dy}{dx}$ where $t = \frac{\pi}{2}$ if

$x = a(t - \sin t)$
$y = a(1 - \cos t)$

15. Find $\frac{dy}{dx}$ when $t = 1$ if $\begin{cases} x = t \log t \\ y = \dfrac{\log t}{t} \end{cases}$

16. Find $\frac{dy}{dx}$ when $t = \frac{\pi}{4}$ if $\begin{cases} x = e^t \cos t \\ y = e^t \sin t \end{cases}$

17. Prove that the function y represented parametrically by the equations

$x = 2t + 3t^2$
$y = t^2 + 3t^3$

satisfies the equation $y = \left(\frac{dy}{dx}\right)^2 + 2\left(\frac{dy}{dx}\right)^3$

18. When $x = 2$ the following equation is true: $x^2 = 2x$.

Does it follow from this that $(x^2)' = (2x)'$ when $x = 2$?

19. Let $y = \sqrt{a^2 - x^2}$. Is it possible to perform term by term differentiation of $x^2 + y^2 = a^2$?

20. Find the derivative $\frac{dy}{dx}$ of the following implicit functions:

(i) $a \cos^2(x + y) = b$ (ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(iii) $\tan y = xy$ (iv) $x^3 + y^3 = a^3$

(v) $xy = \tan^{-1} \frac{x}{y}$ (vi) $x^3 + x^2 y + y^2 = 0$

(vii) $\tan^{-1}(x + y) = x$ (viii) $e^y = x + y$

(ix) $\log x + e^{-\frac{y}{x}} = c$ (x) $\log y + \frac{x}{y} = c$

(xi) $y^3 = \frac{x - y}{x + y}$ (xii) $x^y = y^x$

21. Find at (1, 1) the derivative $\frac{dy}{dx}$ if $2y = 1 + xy^3$.

22. If $ye^y = e^{x+1}$ find $\frac{dy}{dx}$ at $x = 0$ and $y = 1$.

23. If $y^2 = x + \log\frac{y}{x}$ find $\frac{dy}{dx}$ at (1, 1).

24. Find the second derivative of the given functions :

(i) $y = x^8 + 7x^6 - 5x + 4$

(ii) $y = \log(x + \sqrt{a^2 + x^2})$

(iii) $y = e^{x^2}$ (iv) $y = \sin^2 x$

(v) $y = (1 + x^2)\tan^{-1} x$

(vi) $y = (\sin^{-1} x)^2$

(vii) $y = \log \sqrt[3]{1 + x^2}$

25. Show that the function $y=\frac{1}{2}[x^2+2x+2]$ satisfies the differential equation $1+y_1{}^2=2yy_2$.

26. Show that the function $y = \frac{1}{2}x^2e^x$ satisfies the differential equation $y_2 - 2y_1 + y = e^x$.

27. Find y_3 of the function $y = \log(1 + x)$.

28. Find y_4 of the function $y = \sin 2x$.

29. Show that the function $y = e^{-x}\cos x$ satisfies the differential equation $y_4 + 4y = 0$.

30. Find the *n*th derivatives of the functions :

(i) $y = \frac{1}{1-x}$ and (ii) $y = \sqrt{x}$

31. Find the *n*th derivatives of the functions :

(i) $y = \sin x$ (ii) $y = \frac{1}{1+x}$

(iii) $y = \cos 2x$ (iv) $y = \frac{1+x}{1-x}$

(v) $y = e^{-3x}$ (vi) $y = \sin^2 x$

(vii) $y = \log(1 + x)$

(viii) $y = \log(ax + b)$

32. Using Leibniz rule find y_n if

(i) $y = x.e^x$ (ii) $y = x^2e^{-2x}$

(iii) $y = x^3\log x$ (iv) $y = \frac{1+x}{\sqrt{x}}$

(v) $y = (1 - x^2)\cos x$

33. Find $y_n(0)$ if $y = \log\frac{1}{1-x}$

34. In the following problems find $\frac{d^2y}{dx^2}$

(i) $x = \log t$, $y = t^3$ (ii) $x = \tan^{-1} t$, $y = \log(1 + t^2)$

(iii) $x = a\cos t$, $y = a\sin t$ (iv) $x = \cos 2t$, $y = \sin^2 t$

(v) $x = e^{-at}$, $y = e^{at}$ (vi) $x = \log t$, $y = \frac{1}{1-t}$

35. Find $\frac{d^2y}{dx^2}$ if $\left\{\begin{array}{l} x = e^t\cos t \\ y = e^t\sin t\end{array}\right.$

36. In the following examples find $y_3 = \frac{d^3y}{dx^3}$

(i) $x = \sec t$, $y = \tan t$ (ii) $x = e^{-t}$, $y = t^3$

(iii) $x = e^{-t}\cos t$, $y = e^{-t}\sin t$

37. Find $y_2\left(=\frac{d^2y}{dx^2}\right)$ if $x^2 + y^2 = 1$.

38. In the following functions find y_2:

(i) $y^2 = 2px$ (ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

39. Having the equation $y = x + \log y$, find $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}$

40. The equation of motion of a point along the *x*-axis is $x = 100 + 5t - 0.001\,t^3$. Find the velocity and the acceleration of the point for times $t_0 = 0$, $t_1 = 1$ and $t_2 = 10$.

3.15 The Differential of first order

If a function $y = f(x)$ is differentiable in an interval $[a, b]$ then its derivative, as we know, at a point x within $[a, b]$ can be written as

$$\underset{\triangle x\to 0}{Lt}\frac{\triangle y}{\triangle x} = f'(x) \tag{1}$$

This definition also indicates that as $\triangle x \to 0$, the ratio $\frac{\triangle y}{\triangle x}$ approaches a definite number $f'(x)$ and consequently, for all values close to x, we may write

$$\frac{\triangle y}{\triangle x} = f'(x) + \alpha \tag{2}$$

where $\alpha \to 0$ as $\triangle x \to 0$.

Formula (2) may also be expressed as

$$\triangle y = f'(x)\triangle x + \alpha\triangle x \tag{3}$$

Since, in general, $f'(x) \neq 0$ for a constant value of x and a variable $\triangle x \to 0$, the product $f'(x)\triangle x$ can be taken as an infinitesimal of first order relative to $\triangle x$. But the product

$\alpha \triangle x$ is always an infinitesimal of higher order relative to $\triangle x$, because

$$\underset{\Delta x \to 0}{Lt} \frac{\alpha \triangle x}{\triangle x} = \underset{\Delta x \to 0}{Lt} \alpha = 0 \qquad (4)$$

Thus from (3), we find that the increment $\triangle y$ of the function consists of two terms. The first one is regarded as the principal part of the increment and is linear relative to $\triangle x$. This principal part, i.e. $f'(x) \triangle x$ is called the **differential of the function** $y\,(= f(x))$ and is denoted by dy or $df(x)$.

Thus if a function $y = f(x)$ has a derivative $f'(x)$ at the point x, the product of the derivative $f'(x)$ with the increment $\triangle x$ of the independent variable is called the differential of the function and is denoted by the symbol dy as $dy = f'(x) \triangle x$ (5)

3.15 (*a*) *Deductions*

(i) If, in particular, the function be $y = x$ then its derivative is

$$\frac{dy}{dx}\,(= f'(x)) = 1$$

and consequently by (5), we have,

$$dx = 1 . \triangle x = \triangle x$$

This shows that the differential dx of the independent variable x coincides with its increment $\triangle x$. Hence the formula (5) can be written as $dy = f'(x)\,dx$. (6)

This formula is taken as the standard definition for differential of a function y which is a function of single independent variable x.

(ii) The relation (6) also shows that

$$f'(x) = \frac{dy}{dx} \qquad (7)$$

Hence, the derivative $f'(x)$ may be taken as the ratio of the differentials of a function to the differential of the independent variable.

(iii) Again applying (6) to (3) we find that

$$\triangle y = dy + \alpha \triangle x \qquad (8)$$

which indicates that the increment of a function differs from the differential of a function by an infinitesimal quantity $\alpha \triangle x$ of higher order relative to $\triangle x$.

(iv) If $f'(x) \neq 0$, then $\alpha \triangle x$ is an infinitesimal of higher order relative to dy also and we have

$$\underset{\Delta x \to 0}{Lt} \frac{\triangle y}{dy} = 1 + \underset{\Delta x \to 0}{Lt} \frac{\alpha \triangle x}{dy} =$$

$$1 + \underset{\Delta x \to 0}{Lt} \frac{\alpha \triangle x}{f'(x) \triangle x} = 1 + \underset{\Delta x \to 0}{Lt} \frac{\alpha}{f'(x)} \qquad (9)$$

$= 1$, [$\because \alpha \to 0$ as $\triangle x \to 0$].

The expression (9), therefore, helps us to use in approximate calculations (i.e. for small values of $\triangle x$).

$$\triangle y \fallingdotseq dy \qquad (10)$$

or, in an expanded form, we get

$$f(x + \triangle x) - f(x) \fallingdotseq f'(x) \triangle x$$

i.e. $f(x + \triangle x) \fallingdotseq f(x) + f'(x) \triangle x$ (11)

where $y = f(x)$.

3.15 (*b*) *Some illustrative examples*

Let us find the differential dy and the increment $\triangle y$ of the function $y = x^2$.

(i) for arbitrary values of x and $\triangle x$

(ii) for $x = 100$, $\triangle x = .01$

The solutions of (i) and (ii) can be derived as

(i) $\triangle y = (x + \triangle x)^2 - x^2 = 2x \triangle x + (\triangle x)^2$

and $dy = \dfrac{dx^2}{dx} . \triangle x = 2x \triangle x$

In the following figure the distinction between increment of a function $y = x^2$ and its differential has been illustrated.

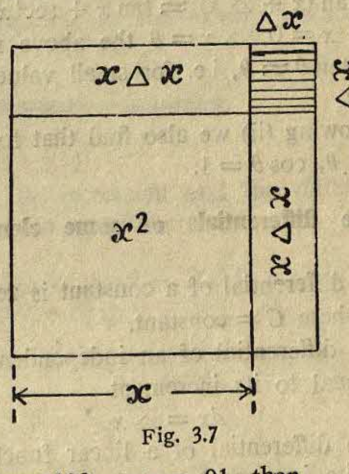


Fig. 3.7

(ii) If $x = 100$, $\triangle x = .01$, then $\triangle y = 2 \times 100 \times .01 + (.01)^2 = 2 + .0001 = 2.0001$ and $dy = 2 \times 100 \times .01 = .2$

These results, in fact, are meant for bringing out numerical differences between the increment $\triangle y$ and the differential dy of a function $y = x^2$.

As another example, let us turn our attention to calculating the approximate value of sin 46° by using formula (11).

To do this, we have from (11)

$$f(x + \triangle x) \approxeq f(x) + f'(x) \triangle x \qquad (12)$$

Let us take $f(x) = \sin x$ then (12) gives

$$\sin (x + \triangle x) \approxeq \sin x + \cos x \,.\, \triangle x \qquad (13)$$

If we put in (13) $x = 45° = \frac{\pi}{4}$ and $\triangle x = 1° = \frac{\pi}{180}$ we get

$$\sin 46° \approxeq \sin \frac{\pi}{4} + \cos \frac{\pi}{4} \cdot \frac{\pi}{180} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\pi}{180} = 0.7071 + 0.7071 \times 0.017 = 0.7194$$

Hence the approximate value of sin 46° = 0.7194.

REMARKS

1. If we put in (13), $x = 0$ and $\triangle x = \theta$ we have $\sin \theta \approxeq \theta$. So for small values of θ', $\sin \theta = \theta$.

2. Similarly if $f(x) = \tan x$, then by (12)

$$\tan (x + \triangle x) \approxeq \tan x + \sec^2 x \triangle x$$

Thus for $x = 0$, $\triangle x = \theta$ the above relation becomes $\tan \theta \approxeq \theta$, i.e. for small values of θ, $\tan \theta = \theta$.

3. Following (ii) we also find that for small values of θ, $\cos \theta = 1$.

3.16 The differentials of some elementary functions

(i) The differential of a constant is zero, i.e. $dC = 0$ where $C =$ constant.

(ii) The differential of an independent variable is equal to its increment

$$dx = \triangle x$$

(iii) The differential of a linear function is equal to its increment

$$d(ax + b) = \triangle(ax + b) = a \triangle x$$

where a and b are constants.

3.17 Properties of a differential

(i) A constant factor may be taken outside the sign of the differential:

$$d\{af(x)\} = adf(x)$$

(ii) The differential of an algebraic sum of a fixed number of functions is equal to the algebraic sum of their differentials:

$$d\{f_1(x) + f_2(x) - f_3(x)\} = df_1(x) + df_2(x) - df_3(x)$$

(iii) The differential of a function is equal to the product of the derivative by the differential of the independent variable:

$$df(x) = f'(x)dx$$

(iv) The differential of a product of two functions is equal to the sum of the products of each of the functions by the differential of the other:

$$d\{f(x)\,\phi(x)\} = f(x)\,d\phi(x) + \phi(x)\,df(x)$$

in short $d(uv) = udv + vdu$, u, v are differentiable functions of x.

(v) The differential of a function is equal to the product of the denominator by the differential of the numerator minus the product of the numerator by the differential of the denominator, the whole expression divided by the denominator squared:

$$d\left\{\frac{f(x)}{\phi(x)}\right\} = \frac{\phi(x)\,df(x) - f(x)\,d\phi(x)}{\{\phi(x)\}^2}, \phi(x) \neq 0$$

in short,

$$d\left\{\frac{u}{v}\right\} = \frac{vdu - udv}{v^2}, \quad v \neq 0\,;\ u, v \text{ are}$$

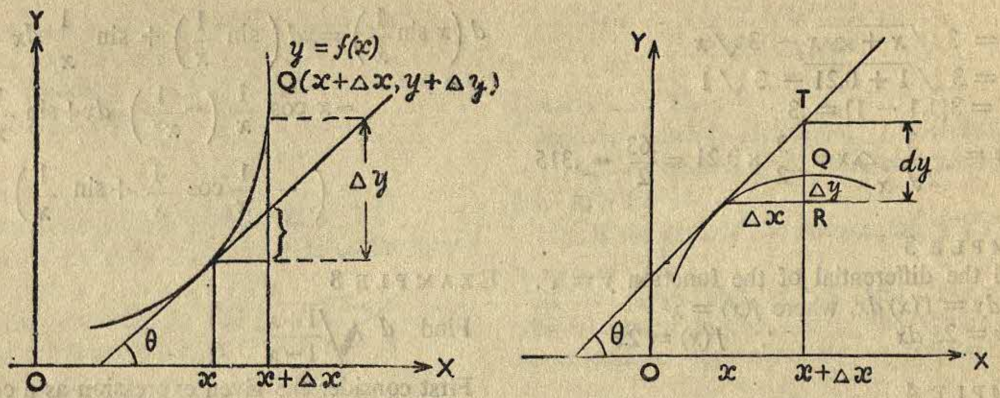
differentiable functions of x.

REMARKS

The problem of finding the differential of a function is equivalent to that of finding the derivative, since, by multiplying the latter into the differential of the independent variable we get the differential of the function. Consequently, most theorems and formulas pertaining to derivatives are also valid for differentials.

3.18 Geometrical significance of the differential

In Fig. 3.8 (i) let the function $y = f(x)$ be represented by the curve. Let θ be the in-



(i) (ii)

Fig. 3.8

clination of the tangent to the curve at $P(x, y)$, and let $\triangle x$ be the increment of the independent variable x. For the increment $\triangle x$, let us suppose that the increment in y is $\triangle y$.

Now, from the triangle PRT, we find

$$RT = PR \tan \theta$$

Since $\tan \theta = f'(x)$, $PR = \triangle x$, we get

$$RT = f'(x) \triangle x$$

but by the definition of a differential

$$dy = f'(x) \triangle x$$

Hence, $dy = RT$

This signifies that the differential of a function $f(x)$, which corresponds to the given values x and $\triangle x$, is equal to the increment in the ordinate of the line tangent to the curve $y = f(x)$ at the given point x.

Also from Fig. (i) it follows directly that

$$QT = \triangle y - dy$$

but it has already been shown that

$$\frac{QT}{RT}\left(\text{i.e. } \frac{QT}{dy}\right) \to 0$$

as $\triangle x \to 0$, i.e. $\dfrac{\triangle y}{dy} \to 1$ as $\triangle x \to 0$.

Hence $\triangle y \approx dy$ for small $\triangle x$.

Although, in Fig. 3.8 (i), the increment $\triangle y$ is shown as greater than dy; it is not always so. For instance in Fig. 3.8 (ii),

$$\triangle y = QR, \quad dy = RT$$

and $\triangle y < dy$

3.19 The mechanical interpretation of a differential

Let $s = f(t)$ be the distance of a rectilinearly moving point from its initial position (t is the time in transit). The increment $\triangle s$ is the distance covered by the point during the time interval $\triangle t$, while the differential $ds = f'(t) \triangle t$ is the distance the point would have covered during time $\triangle t$ if it had maintained the speed $f'(t)$ reached at time t. For an infinitesimal $\triangle t$ the imagined distance ds differs from the true distance $\triangle s$ by an infinitesimal of order higher than $\triangle t$. If the velocity at time t is not equal to zero, then ds yields an approximation of the small displacement of the point.

3.20 Illustrative examples

EXAMPLE 1

Find the increment and the differential of the function $y = 3x^2 - 5x$.

Here,

$$\triangle y = 3(x + \triangle x)^2 - 5(x + \triangle x) - 3x^2 + 5x$$
$$= (6x - 5) \triangle x + 3(\triangle x)^2$$

and $dy = f'(x) \triangle x = \dfrac{dy}{dx} \triangle x = (6x - 5) \triangle x$

EXAMPLE 2

Calculate $\triangle y$ and dy of the function $y = 3\sqrt{x}$ for $x = 1$ and $\triangle x = 0.21$.

Here,

$$\triangle y = 3\sqrt{x+\triangle x} - 3\sqrt{x}$$
$$= 3\sqrt{1+0.21} - 3\sqrt{1}$$
$$= 3[1.1 - 1] = .3$$

and $dy = \dfrac{3}{2\sqrt{x}}\triangle x = \dfrac{3}{2} \times 0.21 = \dfrac{.63}{2} = .315$

EXAMPLE 3

Find the differential of the function $y = x^2$.

Here, $dy = f'(x)\,dx$, where $f(x) = x^2$

$\therefore \quad dy = 2x\,dx \qquad \because \quad f'(x) = 2x$

EXAMPLE 4

Find the differential of the function $y = (1 + x^2)^3$.

Regarding y as the composite function, i.e.

$$y = u^3,\ u = 1 + x^2$$

We get, $dy = 3u^2du$, $du = 2x\,dx$

$\therefore \quad dy = 3u^2.2x\,dx = 3(1 + x^2)^2\,2x\,dx$

$= (6x + 12x^3 + 6x^5)dx$

Alternately,

$$d(1 + x^2)^3 = 3(1 + x^2)^2\,d(1 + x^2) = 3(1 + x^2)^2\,2x\,dx$$

EXAMPLE 5

Find $d\sqrt{a^2 - x^2}$

$$d\sqrt{a^2 - x^2} = d(a^2 - x^2)^{\frac{1}{2}}$$
$$= \tfrac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}\,d(a^2 - x^2)$$
$$= \tfrac{1}{2}(a^2 - x^2)^{-\frac{1}{2}} \times -2x\,dx$$
$$= \frac{-x\,dx}{\sqrt{a^2 - x^2}}$$

EXAMPLE 6

Find the differential of the function

$$(2x^2 + 3x)(x^3 - 2)$$

By 3.16, (4), we have

$$d\{(2x^2 + 3x)(x^3 - 2)\} = (2x^2 + 3x)\,d(x^3 - 2) + (x^3 - 2)\,d(2x^2 + 3x)$$
$$= (2x^2 + 3x)\,3x^2dx + (x^3 - 2)(4x + 3)dx$$
$$= (10x^4 + 12x^3 - 8x - 6)dx$$

EXAMPLE 7

Find $d\left(x \sin\dfrac{1}{x}\right)$ when $x \neq 0$.

$$d\left(x \sin\frac{1}{x}\right) = x\,d\left(\sin\frac{1}{x}\right) + \sin\frac{1}{x}\,dx$$
$$= x\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)dx + \sin\frac{1}{x}\,dx$$
$$= \left(-\frac{1}{x}\cos\frac{1}{x} + \sin\frac{1}{x}\right)dx$$

EXAMPLE 8

Find $d\sqrt{\dfrac{1+x}{1-x}}$

First consider the given expression as a composite function, i.e. $y = \sqrt{u}$, $u = \dfrac{1+x}{1-x}$.

$$d\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\sqrt{\frac{1-x}{1+x}}\;d\left(\frac{1+x}{1-x}\right)$$
$$= \frac{1}{2}\sqrt{\frac{1-x}{1+x}}\;\frac{(1-x)\,dx + (1+x)\,dx}{(1-x)^2},$$

[by 5. §3.16]

$$= \frac{dx}{(1-x)\sqrt{1-x^2}}.$$

EXAMPLE 9

Find $d\log\dfrac{1+x}{1-x}$.

$$d\log\frac{1+x}{1-x} = d\log(1 + x) - d\log(1 - x)$$
$$= \frac{1}{1+x}\,d(1+x) - \frac{1}{1-x}\,d(1-x)$$
$$= \frac{dx}{1+x} + \frac{dx}{1-x} = \frac{2dx}{1-x^2}$$

EXAMPLE 10

Find log 101 without using tables.

The increment $\triangle(\log x) \approx d(\log x)$

But $d(\log x) = \dfrac{1}{x}\triangle x$

Now for $x = 100$ and $\triangle x = 1$ we obtain

$$\triangle(\log 100) \approx \frac{1}{100}.1 = .01$$

Again, $\log(x + \triangle x) = y + \triangle y = \log x + \triangle \log x$

and for $x = 100$ and $\triangle x = 1$, we have

$$\log(100+1) = \log 100 + \triangle(\log 100)$$

$$\text{or, } \log 101 = 2 + .01 = 2.01$$

which coincides with the tabular value of natural logarithm.

EXAMPLE 11

Extract the square root of 3654.

It is necessary to find the value of the function $f(x) = \sqrt{x}$ for $x = 3654$.

Now for the function $f(x) = \sqrt{x}$, the approximate relations given in (11) § 3.14 is

$$\sqrt{(x+\triangle x)} \fallingdotseq \sqrt{x} + \frac{d\sqrt{x}}{dx} \cdot \triangle x$$

To calculate the above root we choose, $x = 3600$ and $\triangle x = 54$. This gives

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{3600}} = \frac{1}{120}$$

$$\therefore\ \sqrt{3654} \fallingdotseq \sqrt{3600} + 54 = \sqrt{3600} + \frac{1}{120} \cdot 54 = 60 + \frac{54}{120} = 60.45 \text{ (approx.)}$$

EXAMPLE 12

Without using table, find the value of tan 46°.

Put $f(x) = \tan x$, $x = 45°$, $\triangle x = 1° = 0.0175$ radian.

Then we have

$$f'(x) = \sec^2 x, \text{ i.e. } f'(45°) = \frac{1}{\cos^2 45°} = 2$$

Hence

$$\tan(45° + 1°) \fallingdotseq \tan 45° + f'(45°) \times \triangle x$$

$$\therefore\ \tan 46° \fallingdotseq 1 + 2 \times 0.0175 = 1.0350 \text{ (approx.)}$$

EXAMPLE 13

By how much (approximately) does the side of a square change if its area increases from 9 m^2 to 9.1 m^2?

If x is the area of the square and y its side, then $y = \sqrt{x}$.

It is given that $x = 9$ and $\triangle x = 0.1$. The increment in the side of the square may be calculated approximately as follows:

$$\triangle y \fallingdotseq dy = \frac{dy}{dx}\triangle x = \frac{1}{2\sqrt{9}} \times 0.1 = 0.016 \text{ m}.$$

So the side of the square changes from 3 m to 3.016 m.

Exercise 3.3

1. Find the increment $\triangle y$ and the differential dy of the function $y = 5x + x^2$ for $x = 2$ and $\triangle x = 0.001$.
2. Find $d(1 - x^3)$ for $x = 1$ and $\triangle x = -\frac{1}{3}$.
3. Find the differential of the function $$y = \frac{2}{\sqrt{x}} \text{ for } x = 9 \text{ and } \triangle x = -0.01$$
4. Calculate the differential of the function $$y = \tan x \text{ for } x = \frac{\pi}{3} \text{ and } \triangle x = \frac{\pi}{180}.$$
5. Find the approximate value of sin 31°.
6. Calculate approximately the values of:

(i) cos 61° (ii) tan 44° (iii) log 0.9
(iv) $\sqrt{1.2}$ (v) $\tan^{-1}(1.05)$

INTEGRAL CALCULUS

4 Integration of Functions

4.1 Definition and the basic properties of integration

We are already familiar with one operation on functions, namely, 'differentiation'. We know how to obtain the derivative (as a function) of a given function. Let us now look at the inverse problem, i.e. how to obtain the function, given the derivative of a function. This leads to the operation called integration. The process of integration can, however, be defined in two ways. Either as the inverse of differentiation which we have mentioned above or as the limit of a summation. For the present, we shall consider integration as the inverse operation of differentiation. After some preliminary chapters we shall show that these two definitions are equivalent.

Suppose $f(x)$ be a given function of the variable x and if another function $\phi(x)$ can be found such that the differential coefficient of the function $\phi(x)$ with respect to x is equal to $f(x)$, then $\phi(x)$ is called **the integral** or an **indefinite integral** of $f(x)$ with respect to x. The process of obtaining the function $\phi(x)$ is known as the **integration process.** Symbolically, this can be written as follows:

$$\text{If} \quad \frac{d}{dx}\phi(x) = f(x),$$

then we write $\phi(x) = \int f(x)dx.$

$\int f(x)\,dx$ is called an indefinite integral of $f(x)$ with respect to x. The sign $\int$ is called the **integral sign.** Thus the operation of integration is indicated by writing the above sign before the product of the function $f(x)$ and the differential dx of the variable x. The function $f(x)$ is called the **integrand.** Since, here, the limits within which the function $f(x)$ is to be integrated with respect to x are not specified, the integral is called an **indefinite integral.** On the other hand, if the limits are prescribed then the integral is called a **definite integral.** These will be discussed further in later chapters.

4.2 Constant of integration

By definition, if $\phi(x)$ be the integral of some function $f(x)$ then $\frac{d}{dx}\phi(x) = f(x)$.

Since the derivative of any constant c is zero we can write the above relation also in the form

$$\frac{d}{dx}\left\{ \phi(x) + c \right\} = f(x)$$

where c is an arbitrary constant. So the integral of $f(x)$ with respect to x becomes $\int f(x)dx = \phi(x) + c$. If the arbitrary constant $c = 0$, the indefinite integral or simply the integral of the function $f(x)$ with respect to x reduces to $\int f(x)dx = \phi(x)$. Thus we find that the general value of the indefinite integral $\int f(x)dx$ is $\phi(x) + c$. This constant c is called the **constant of integration.** For the sake of simplicity and convenience, the arbitrary constant of integration has been dropped in all the examples considered here but it is always understood to be present.

4.3 Some properties of integrals

(i) The operation of integration is commutative with regard to a constant.

Symbolically this can be written as

$$\int Af(x)dx = A\int f(x)dx$$

P ROOF

Suppose $f(x)$ and $\phi(x)$ be two functions of x such that $\dfrac{d}{dx}\left\{\phi(x)\right\} = Af(x)$, A being a constant.

$$\text{Then } \phi(x) = \int Af(x)dx$$

$$\text{Again } \frac{d}{dx}\left\{\frac{\phi(x)}{A}\right\} = f(x)$$

$$\text{So } \quad \frac{\phi(x)}{A} = \int f(x)dx$$

$$\text{Or } \phi(x) = A\int f(x)dx$$

$$\text{Thus } \int Af(x)dx = A\int f(x)dx.$$

(ii) Integral of the sum or difference of any finite number of functions is equal to the sum or difference of the integrals of those functions.

Symbolically this can be written as :

$$\int\left\{f(x) + \phi(x) - \psi(x) + \dots\dots\right\}dx$$

$$= \int f(x)dx + \int \phi(x)dx - \int \psi(x)dx + \dots\dots\dots+\dots\dots\dots$$

P ROOF

Suppose $\chi(x)$ be a function of the variable x such that

$$\frac{d}{dx}\chi(x) = f(x) + \phi(x) - \psi(x) + \dots\dots$$

where $f(x)$, $\phi(x)$, $\psi(x)$..., etc. are each functions of the variable x. Again let us suppose that these functions are the derivatives of functions $\chi_1(x)$, $\chi_2(x)$, $\chi_3(x)$, etc. Then we find

$$\frac{d}{dx}\chi(x) = \frac{d}{dx}\chi_1(x) + \frac{d}{dx}\chi_2(x) - \frac{d}{dx}\chi_3(x) + ..$$

$$\text{or } \chi(x) = \chi_1(x) + \chi_2(x) - \chi_3(x) + \dots\dots\dots\dots$$

Moreover, since

$$\frac{d}{dx}\chi(x) = f(x) + \phi(x) - \psi(x) + \dots\dots$$

We get from the definition of integration

$$\chi(x) = \int [f(x) + \phi(x) - \psi(x) + \dots\dots]dx$$

$$\therefore \ \chi_1(x) + \chi_2(x) - \chi_3(x) + \dots = \int [f(x) + \phi(x) - \psi(x) + \dots]dx$$

Again $x_1(x)$, $x_2(x)$, etc. are the integrals $\int f(x)dx$, $\int \phi(x)dx$, etc. So we find

$$\int [f(x) + \phi(x) - \psi(x) + \dots\dots]dx$$

$$= \int f(x)dx + \int \phi(x)dx - \int \psi(x)dx + \dots\dots$$

Hence the result.

4.4 Fundamental integrals

The operation of integration being the inverse operation of differentiation, integrals of many elementary functions can be determined very easily.

We know from Differential Calculus that

$$\text{(i)} \quad \frac{d}{dx}x^{n+1} = (n+1)x^n$$

$$\text{So } \quad \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \ (n \neq -1)$$

$$\therefore \int x^n dx = \frac{x^{n+1}}{n+1}$$

In particular if $n = 0$ then $\int dx = x.$

(ii) $\dfrac{d}{dx}\,(\log x) = \dfrac{1}{x}$

$$\therefore \quad \int \frac{dx}{x} = \log x$$

(iii) $\dfrac{d}{dx}\left(e^{mx}\right) = me^{mx}$

So $\dfrac{d}{dx}\left(\dfrac{e^{mx}}{m}\right) = e^{mx}$

$$\therefore \quad \int e^{mx}dx = \frac{e^{mx}}{m}$$

In particular if $m = 1$ then $\int e^{x}dx = e^{x}.$

(iv) $\dfrac{d}{dx}\left(a^{x}\right) = a^{x}\log_e a$

So $\dfrac{d}{dx}\left(\dfrac{a^{x}}{\log_e a}\right) = a^{x}$

$$\therefore \quad \int a^{x}dx = \frac{a^{x}}{\log_e a}$$

(v) $\dfrac{d}{dx}\,(\sin mx) = m\cos mx$

So $\dfrac{d}{dx}\left(\dfrac{\sin mx}{m}\right) = \cos mx$

$$\therefore \quad \int \cos mx\,dx = \frac{\sin mx}{m}$$

In particular if $m = 1$ then $\int \cos x\,dx = \sin x.$

(vi) $\dfrac{d}{dx}\left(\cos mx\right) = -m\sin mx$

So $\dfrac{d}{dx}\left(\dfrac{\cos mx}{-m}\right) = \sin mx$

$$\therefore \quad \int \sin mx\,dx = -\frac{\cos mx}{m}$$

In particular if $m = 1$ then $\int \sin x\,dx =$

$= \cos x.$

(vii) $\dfrac{d}{dx}\,(\tan x) = \sec^2 x$

$$\therefore \quad \int \sec^2 x\,dx = \tan x$$

(viii) $\dfrac{d}{dx}\,(\cot x) = -\operatorname{cosec}^2 x$

$$\therefore \quad \int \operatorname{cosec}^2 x\,dx = -\cot x$$

(ix) $\dfrac{d}{dx}\,(\sec x) = \sec x\tan x$

$$\therefore \quad \int \sec x\tan x\,dx = \sec x$$

(x) $\dfrac{d}{dx}\,(\operatorname{cosec} x) = -\operatorname{cosec} x\cot x$

$$\therefore \quad \int \operatorname{cosec} x\cot x\,dx = -\operatorname{cosec} x.$$

4.5 Illustrative examples

EXAMPLE 1

Integrate $\int \sqrt{x}\left(x^3+\dfrac{3}{x^2}\right)dx$

$$\int \sqrt{x}\;\; x^3+\frac{3}{x^2}\;\; dx = \int x^{\frac{7}{2}}\,dx + 3\int x^{-\frac{3}{2}}\,dx + C$$

$$= \frac{2}{9}\,x^{\frac{9}{2}} - 6x^{-\frac{1}{2}} + C$$

where C is the constant of integration.

EXAMPLE 2

$$\int \cos 2x\cos 3x\,dx$$

$\cos 2x\cos 3x = \tfrac{1}{2}[\cos 5x + \cos x]$

$$\therefore \quad \int \cos 2x\cos 3x\,dx = \tfrac{1}{2}\int \cos 5x\,dx + \tfrac{1}{2}\int \cos x\,dx + C$$

$$= \frac{\sin 5x}{10} + \frac{\sin x}{2} + C$$

EXAMPLE 3

$$\int \sin^2 x \cos^2 x \, dx$$

$$= \tfrac{1}{4} \int 4 \sin^2 x \cos^2 x \, dx = \tfrac{1}{4} \int \sin^2 2x \, dx$$

$$= \tfrac{1}{8} \int 2 \sin^2 2x \, dx = \tfrac{1}{8} \int (1 - \cos 4x) dx =$$

$$\tfrac{1}{8} \int dx - \tfrac{1}{8} \int \cos 4x \, dx$$

$$= \frac{x}{8} - \frac{\sin 4x}{32} + C$$

Exercise 4.1

1. Integrate the following:

(i) $\int \frac{5 (x-3)^2}{x \sqrt{x}} \, dx$ (ii) $\int \frac{\tan x}{\cot x} \, dx$

(iii) $\int \frac{e^{3x}+e^{5x}}{e^{x}+e^{-x}} \, dx$ (iv) $\int \sqrt{1 + \sin x \, dx}$

(v) $\int \cos^3 x \, dx$

(vi) $\int \sin x \sin 2x \sin 3x \, dx$

(vii) $\int \frac{dx}{1+\sin x}$ (viii) $\int \frac{x^3-6x+9}{x+3} \, dx$

(ix) $\int \left\{ \frac{\alpha_3^{x}+\alpha^{x}+\alpha^{x}}{\alpha_4^{x}} \right\} dx$

(x) $\int \frac{dx}{\sin^2 x \cos^2 x}$

5 Methods of Integration

It is often found that integrals are not given in standard forms. To reduce them to standard forms we have to apply different methods of integration. In this chapter we shall discuss these methods.

5.1 Method of substitution and the change of variable: working rule

Suppose we are to integrate the function $f(x)$ with respect to x.

Then $\int f(x)\,dx$ is to be evaluated.

Let us substitute $x = \phi(z)$.

Then $dx = \phi'(z)\,dz$

So $\int f(x)\,dx = \int f[\phi(z)]\ \phi'(z)\,dz$

Thus by the substitution $x = \phi(z)$, we have converted the given integral into another integral where the variable is z. Now integrate it and replace z by $\phi^{-1}(x)$ in the final result. This process is known as **integrating by substitution method.**

5.2 Illustrative examples

Example 1

Integrate $\displaystyle\int \frac{\cos(\log x)}{x}\,dx$

Put $\log x = z$.

So $\dfrac{dz}{dx} = \dfrac{1}{x}$

$$\int \frac{\cos(\log x)}{x}\,dx = \int \cos z\,dz = \sin z + C$$

$$= \sin(\log x) + C$$

Example 2

$$\int \sqrt{\frac{a+x}{a-x}}\,dx$$

Put $x = a\cos 2\theta$.

$\therefore\ \dfrac{dx}{d\theta} = -2a\sin 2\theta$.

$$\text{So}\ \int \sqrt{\frac{a+x}{a-x}}\,dx = \int \sqrt{\frac{a+a\cos 2\theta}{a-a\cos 2\theta}}\,(-2a\sin 2\theta)\,d\theta.$$

$$= -4a\int \cos^2\theta\,d\theta = -2a\int(1+\cos 2\theta)\,d\theta.$$

$$= -2a\left\{\theta + \frac{\sin 2\theta}{2}\right\} + C$$

$$= -a\cos^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2} + C$$

Example 3

Integrate $\displaystyle\int \frac{(1+\cos x)}{\sqrt[3]{x+\sin x}}\,dx$

Put $x + \sin x = z$

Then $\dfrac{dz}{dx} = 1 + \cos x$

$$\text{So} \int \frac{(1+\cos x)}{\sqrt[3]{x+\sin x}}\,dx = \int z^{-\frac{1}{3}}\,dz = \frac{3}{2}\,z^{\frac{2}{3}} + C$$

$$= \tfrac{3}{2}(x+\sin x)^{\frac{2}{3}} + C.$$

5.3 Some important standard integrals

(i) $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

PROOF

Put $x = a\tan\theta$, then $dx = a\sec^2\theta\, d\theta$.

$$\text{So } \int\frac{dx}{x^2+a^2}=\int\frac{a\sec^2\theta\, d\theta}{a^2(1+\tan^2\theta)}=\frac{1}{a}\int d\theta=\frac{\theta}{a}$$

$$=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

(ii) $$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\frac{x-a}{x+a}\left(x>a\right)$$

PROOF

The integrand

$$\frac{1}{x^2-a^2}=\frac{1}{2a}\left[\frac{1}{x-a}-\frac{1}{x+a}\right]$$

$$\text{So } \int\frac{dx}{x^2-a^2}=\frac{1}{2a}\int\frac{dx}{x-a}-\frac{1}{2a}\int\frac{dx}{x+a}$$

To integrate $\int\frac{dx}{x-a}$ let us put $x-a=z$.

Then $dx = dz$.

$$\text{So } \int\frac{dx}{x-a}=\int\frac{dz}{z}=\log z=\log(x-a).$$

By similar substitution it can be shown that

$$\int\frac{dx}{x+a}=\log(x+a)$$

$$\therefore \int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log(x-a)-\frac{1}{2a}\log(x+a)$$

$$=\frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$$

(iii) $$\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$$

The proof is the same as (ii) above.

(iv) $$\int\frac{dx}{\sqrt{x^2\pm a^2}}=\log\left(x+\sqrt{x^2\pm a^2}\right)$$

PROOF

Put $z = x+\sqrt{x^2\pm a^2}$, then

$$dz=\frac{[x+\sqrt{x^2\pm a^2}]\,dx}{\sqrt{x^2\pm a^2}}=\frac{z}{\sqrt{x^2\pm a^2}}\,dx$$

$$\text{So } \frac{dz}{z}=\frac{dx}{\sqrt{x^2\pm a^2}}$$

$$\therefore \int\frac{dx}{\sqrt{x^2\pm a^2}}=\int\frac{dz}{z}=\log z$$

$$=\log\left\{x+\sqrt{x^2\pm a^2}\right\}$$

(v) $$\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)$$

Put $x = a\sin\theta$.

$\therefore\ dx = a\cos\theta\, d\theta$.

$$\text{So } \int\frac{dx}{\sqrt{a^2-x^2}}=\int d\theta=\theta=\sin^{-1}\left(\frac{x}{a}\right)$$

(vi) Integrate $\int\frac{dx}{ax^2+bx+c}$

The given integral can be written as

$$\frac{1}{a}\int\frac{dx}{x^2+\frac{b}{a}x+\frac{c}{a}}=\frac{1}{a}\int\frac{dx}{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}$$

$$\text{or } \frac{1}{a}\int\frac{dx}{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}}$$

according as $4ac-b^2>0$ or <0.

If $4ac-b^2>0$, then the given integral becomes

$$\frac{1}{a}\int\frac{dx}{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}$$

$$=\frac{1}{a}\int\frac{dx}{\left(x+\frac{b}{2a}\right)^2+\left(\frac{\sqrt{4ac-b^2}}{2a}\right)^2}$$

Now put $x+\frac{b}{2a}=z$, then the above integral reduces to $\frac{1}{a}\int\frac{dz}{z^2+K^2}$ where $K=\frac{\sqrt{4ac-b^2}}{2a}$

(constant)

$$=\frac{1}{Ka}\tan^{-1}\left(\frac{z}{K}\right)=\frac{2}{\sqrt{4ac-b^2}}$$

$$\tan^{-1}\left\{\frac{2ax+b}{\sqrt{4ac-b^2}}\right\}$$

Now, if $4ac-b^2<0$, then the integral becomes

$$\frac{1}{a}\int\frac{dx}{\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ac}{4a^2}\right)}$$

Now let us put $x+\dfrac{b}{2a}=z$, then the above integral reduces to $\dfrac{1}{a}\displaystyle\int\frac{dz}{z^2-\lambda^2}$, where $\lambda = \dfrac{\sqrt{b^2-4ac}}{2a}$ (constant).

$$=\frac{1}{2a\lambda}\log\left\{\frac{z-\lambda}{z+\lambda}\right\}=\frac{1}{\sqrt{b^2-4ac}}\log\left[\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right]$$

(vii) Integrate $\displaystyle\int\frac{\alpha x+\beta}{ax^2+bx+c}\,dx$

The integrand $\dfrac{\alpha x+\beta}{ax^2+bx+c}$

$$=\frac{\alpha}{2a}\frac{\left\{2ax+b+\frac{2a\beta}{\alpha}-b\right\}}{ax^2+bx+c}$$

So
$$\int\frac{\alpha x+\beta}{ax^2+bx+c}\,dx=\frac{\alpha}{2a}\int\frac{(2ax+b)\,dx}{ax^2+bx+c}+\frac{\alpha}{2a}\left(\frac{2a\beta-b\alpha}{\alpha}\right)\int\frac{dx}{ax^2+bx+c}$$

$$=\frac{\alpha}{2a}\int\frac{d(ax^2+bx+c)}{ax^2+bx+c}+\frac{2a\beta-b\alpha}{2a}\int\frac{dx}{ax^2+bx+c}$$

The first integral $=\log(ax^2+bx+c)$ and the second integral can be evaluated as in (vi).

(viii) Integrate $\displaystyle\int\frac{dx}{\sqrt{ax^2+bx+c}}$

$$\int\frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{x^2+\frac{b\,x}{a}+\frac{c}{a}}}$$

$$=\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}}$$

if $4ac-b^2>0$, and if $4ac-b^2<0$ then the integral becomes

$$\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}}}$$

Both of these two integrals can be reduced to the form discussed in (iv) by the simple substitution $x+\dfrac{b}{2a}=z$. Hence the given integral can be evaluated.

5.4 Integration by parts

It is known from differential calculus that if u and v_1 are two differentiable functions of x, then

$$\frac{d}{dx}\left(uv_1\right)=\frac{du}{dx}\,v_1+u\frac{dv_1}{dx}$$

Now integrating both sides with respect to x, we find

$$uv_1=\int\left(\frac{du}{dx}v_1\right)dx+\int\left(u\frac{dv_1}{dx}\right)dx$$

or
$$\int\left(u\frac{dv_1}{dx}\right)dx=uv_1-\int\left(\frac{du}{dx}v_1\right)dx$$

Let us suppose $\dfrac{dv_1}{dx}=v$ then $v_1=\displaystyle\int v\,dx$

Now writing v for $\dfrac{dv_1}{dx}$ and $\displaystyle\int v\,dx$ for v_1, we find

$$\int(uv)\,dx=u\int v\,dx-\int\left\{\left(\int v\,dx\right)\frac{du}{dx}\right\}dx$$

This is the formula for the integration of a function which can be written as the product of two functions. According to this rule, the integral of the product of two functions = first function × integral of the second − integral of (differential coefficient of the first × integral of the second).

5.5 Illustrative examples

EXAMPLE 1

Integrate $\int x^n \log x \, dx$

Integrating by parts we find

$$\int x^n \log x \, dx = \log x \int x^n \, dx - \int \left\{ \int x^n \, dx \right\} d(\log x)$$

$$= \log x \left(\frac{x^{n+1}}{n+1} \right) - \int \frac{x^{n+1}}{(n+1)} \cdot \frac{1}{x} \, dx$$

$$= \log x \left(\frac{x^{n+1}}{n+1} \right) - \frac{x^{n+1}}{(n+1)^2} + C$$

EXAMPLE 2

Integrate $\int \sqrt{a^2 - x^2} \, dx$

Integrating by parts

$$\int \sqrt{a^2 - x^2} \, dx = x \sqrt{a^2 - x^2} + \int \frac{x^2 \, dx}{\sqrt{a^2 - x^2}}$$

$$= x \sqrt{a^2 - x^2} + \int \frac{[a^2 - (a^2 - x^2)] \, dx}{\sqrt{a^2 - x^2}}$$

$$= x \sqrt{a^2 - x^2} + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} - \int \sqrt{a^2 - x^2} \, dx + C$$

$$\therefore 2 \int \sqrt{a^2 - x^2} \, dx = x \sqrt{a^2 - x^2} + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} + C$$

$$= x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore \int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

EXAMPLE 3

Integrate $\int \frac{x e^x}{(x+1)^2} \, dx$

$$\int \frac{x e^x}{(x+1)^2} \, dx = \int \frac{\{(x+1) e^x - e^x\}}{(x+1)^2} \, dx$$

$$= \int \frac{e^x}{x+1} \, dx - \int \frac{e^x}{(x+1)^2} \, dx$$

Now integrating the first integral by parts we get

$$\int \frac{e^x}{x+1} \, dx = \frac{1}{x+1} e^x + \int \frac{e^x}{(x+1)^2} \, dx + C$$

So $$\int \frac{x e^x}{(x+1)^2} \, dx = \frac{e^x}{(x+1)} + \int \frac{e^x}{(x+1)^2} \, dx - \int \frac{e^x}{(x+1)^2} \, dx + C$$

$$= \frac{e^x}{x+1} + C$$

5.6 Standard integrals

(i) $$\int e^{ax} \cos bx \, dx = e^{ax} \left(\frac{a \cos bx + b \sin bx}{a^2 + b^2} \right) = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos \left\{ bx - \tan^{-1} \frac{b}{a} \right\}$$

PROOF

Integrating by parts we find

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax} \sin bx}{b} - \frac{a}{b} \int e^{ax} \sin bx \, dx$$

Again integrating by parts the integral on the right hand side we find

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}\sin bx}{b} - \frac{a}{b}\left[-\frac{\cos bx}{b}e^{ax} + \int\frac{\cos bx}{b}\,ae^{ax}\,dx\right]$$

$$\therefore \int e^{ax}\cos bx\,dx = \frac{e^{ax}\sin bx}{b} + \frac{a}{b^2}\,e^{ax}\cos bx - \frac{a^2}{b^2}\int\cos bx\,e^{ax}\,dx$$

or $\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cos bx\,dx$

$$= \frac{ae^{ax}\cos bx + be^{ax}\sin bx}{b^2}$$

$$\therefore \int e^{ax}\cos bx\,dx = e^{ax}\left(\frac{a\cos bx + b\sin bx}{a^2+b^2}\right)$$

Let us suppose $a = r\cos\theta$, $b = r\sin\theta$, then $r^2=a^2+b^2$, $\theta=\tan^{-1}\frac{b}{a}$. Putting these values in the right hand side of the above integral we find

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left\{bx - \tan^{-1}\frac{b}{a}\right\}$$

(ii) $\int e^{ax}\sin bx\,dx = e^{ax}\left(\frac{a\sin bx - b\cos bx}{a^2+b^2}\right)$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left\{bx - \tan^{-1}\frac{b}{a}\right\}$$

This can be proved exactly in the same way.

(iii) $\int\sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)$

Integrating by parts

$$\int\sqrt{x^2+a^2}\,dx = x\sqrt{x^2+a^2} - \int\frac{x^2\,dx}{\sqrt{x^2+a^2}} \qquad \text{(A)}$$

Again $\int\sqrt{x^2+a^2}\,dx = \int\frac{x^2+a^2\;dx}{\sqrt{x^2+a^2}}$

$$= \int\frac{x^2dx}{\sqrt{x^2+a^2}} + a^2\int\frac{dx}{\sqrt{x^2+a^2}} \qquad \text{(B)}$$

Now adding (A) and (B) and dividing by 2, we find

$$\int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\int\frac{dx}{\sqrt{x^2+a^2}}$$

or $\int\sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log(x+\sqrt{x^2+a^2})$

Vide, section 5.3(iv).

(iv) $\int\sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log(x+\sqrt{x^2-a^2})$

Integrating by parts

$$\int\sqrt{x^2-a^2}\,dx = x\sqrt{x^2-a^2} - \int\frac{x^2\,dx}{\sqrt{x^2-a^2}}$$

$$= x\sqrt{x^2-a^2} - \int\frac{(x^2-a^2)+a^2\;dx}{\sqrt{x^2-a^2}}$$

$$= x\sqrt{x^2-a^2} - \int\sqrt{x^2-a^2}\,dx - a^2\int\frac{dx}{\sqrt{x^2-a^2}}$$

shifting $\int\sqrt{x^2-a^2}\,dx$ to the left hand side and then dividing by 2 we get

$$\int\sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}$$

$\log(x + \sqrt{x^2 - a^2})$
(By Section 5.3 (iv))

Exercise 5.1

Integrate the following:

1. $\int \frac{x\,dx}{x^4+1}$

2. $\int (x+1)(x^2+2x+3)^{\frac{n}{2}}\,dx$

3. $\int x\sqrt{\frac{1+x}{1-x}}\,dx$ 4. $\int \frac{(x+1)}{3+2x+x^2}\,dx$

5. $\int \frac{e^x\,dx}{e^{2x}+2e^x+5}$ 6. $\int \frac{dx}{\sqrt{2ax-x^2}}$

7. $\int \frac{x^2-x+1}{x^2+x+1}\,dx$

8. $\int \frac{dx}{(2x+1)\sqrt{4x+3}}$ 9. $\int xe^x\,dx$

10. $\int \tan^{-1}x\,dx$

11. $\int e^{2x}\sin 3x\cos x\,dx$

12. $\int \left\{\frac{1}{\log x} - \frac{1}{(\log x)^2}\right\} dx$

13. $\int \sin^{-1} \frac{\sqrt{x}}{x+a}\,dx$

14. $\int e^x \sin^2 x\,dx$

15. $\int \cos 2x \log(1+\tan x)\,dx$

6 Definite Integrals

SO FAR we have considered integration as the inverse of differentiation. Now we shall consider the operation of **integration as the limit of some summation.** In fact the subject of Integral Calculus originated from the necessity of calculating area under curves and later it was found that it can be considered as the inverse operation of differentiation.

6.1 Area under any curve bounded by the two ordinates and x-axis

Let us suppose that the equation to the continuous curve PQ be $y = f(x)$. This curve is continuous in the interval (ab), $b > a$. AC and BD be the two ordinates corresponding to the abscissa $x = a$ and $x = b$. $OA = a$ and $OB = b$. So $AB = b - a$. The length $b - a$ has been divided into n equal parts, each of length h, such that $b - a = nh$ or $b = a + nh$. $A_1, A_2, A_3 \dots\ A_{n-1}$ are the $(n - 1)$ points which divide the length $b - a$ into n equal intervals. The abscissae of these points are $a + h,\ a + 2h, \dots\dots\dots a + (n - 1)h$. From these points, $(n - 1)$ ordinates are erected so as to meet the curve PQ at $(n - 1)$ points. Let us complete the set of inner rectangles $ACC_1A_1, \dots\dots\dots$ and the outer rectangles. Let S denote the area enclosed between the curve $y = f(x)$, the two ordinates $x = a$, $x = b$ and the x-axis. Let S_1 and S_2 denote the sum of the areas of the inner and outer rectangles respectively. Then $S_1 < S < S_2$. Now S_1 = sum of the areas of the inner rectangles

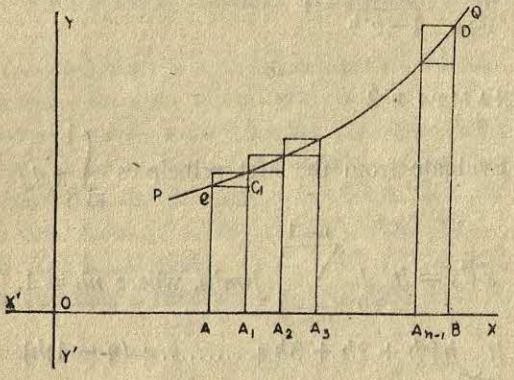


Fig. 6.1

$$= hf(a) + hf(a + h) + hf(a + 2h) + \dots\dots\dots + hf(a + (n - 1)h)$$

$$= h\sum_{m=0}^{n-1} f(a+mh)$$

Similarly

$$S_2 = \text{sum of the areas of the outer rectangles}$$
$$= hf(a + h) + hf(a + 2h) + hf(a + 3h) + \dots\dots + \dots\dots\dots + hf(a + nh)$$

$$= h\sum_{m=1}^{n} f(a+mh)$$

$$= h\sum_{m=0}^{n-1} f(a + mh) - hf(a) + hf(b)$$

Now, let us suppose that the number of subdivisions increases indefinitely. Then, as a consequence, the length of the sub-intervals h will diminish indefinitely. Thus as $n \to \infty$, $h \to 0$.

Therefore,

$$S_1 \to \mathop{lt}_{h\to 0} h \sum_{m=0}^{n-1} f(a+mh)$$

and $$S_2 \to \mathop{lt}_{h\to 0} h \sum_{m=0}^{n-1} f(a+mh)$$

as $f(a)$ and $f(b)$ are finite quantities.
Again, we have always

$$S_1 < S < S_2$$

$$\therefore \quad S = \mathop{lt}_{h\to 0} h \sum_{m=0}^{n-1} f(a+mh)$$

The limit of the sum

$$\mathop{lt}_{h\to 0} h \sum_{m=0}^{n-1} f(a+mh)$$

is denoted by the integral $\int_a^b f(x)\,dx$. a and b are known as the **lower and upper limits of integration.** Thus we find that the integral $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)^a$ bounded by the two ordinates $x = a$ and $x = b$ and the x-axis. Here the function $f(x)$ has to be integrated with respect to x within the limit a to b. This integral is called a **definite integral.**

We have seen in the last article that a definite integral can always be expressed as the limit of a sum. So it can be evaluated just by determining the limit of the corresponding summation. This method of evaluating a definite integral is called **evaluating from the first principles** or **evaluating by the method of summation.**

6.2 Illustrative examples

EXAMPLE 1

Evaluate from the first principles $\int_a^b e^{-x}dx$.

From definition $\int_a^b e^{-x}dx = \mathop{lt}_{h\to 0} h \sum_{m=0}^{n-1} f(a+mh)$,

where $nh = b - a$.

$$\text{So } \int_a^b e^{-x}dx = \mathop{lt}_{h\to 0} h \left[e^{-a} + e^{-(a+h)} + e^{-(a+2h)} + \ldots\ldots + e^{-(a+(n-1)h)} \right]$$

$$= \mathop{lt}_{h\to 0} he^{-a} \left[1 + e^{-h} + e^{-2h} + \ldots\ldots + e^{-(n-1)h} \right]$$

$$= \mathop{lt}_{h\to 0} e^{-a} \left(1 - e^{-nh} \right) \frac{h}{1-e^{-h}}$$

$$= \mathop{lt}_{h\to 0} e^{-a} \left[1 - e^{-(b-a)} \right] \frac{h}{1-e^{-h}}$$

$$= (e^{-a} - e^{-b}) \times 1 = e^{-a} - e^{-b}$$

as $\mathop{lt}_{h\to 0} \dfrac{h}{1-e^{-h}} = 1$

EXAMPLE 2

Evaluate from the first principles $\int_0^1 x\,dx$

$$\int_0^1 x\,dx = \mathop{lt}_{h\to 0} h \sum_{m=0}^{n-1} f(mh), \text{ where } nh = 1.$$

$$= \mathop{lt}_{h\to 0} h\,[\,h + 2h + 3h + \ldots\ldots\ldots + (n-1)h]$$

$$= \mathop{lt}_{h\to 0} \frac{n(n-1)h^2}{2}.$$

$$= \mathop{lt}_{h\to 0} \frac{nh(nh-h)}{2} = \mathop{lt}_{h\to 0} \frac{1.(1-h)}{2} = \frac{1}{2} \quad \text{(Ans).}$$

EXAMPLE 3

Evaluate by the method of summation

$$\int_0^1 (\alpha x^2 + \beta)\, dx$$

Now $\displaystyle\int_0^1 (\alpha x^2 + \beta)\, dx = \underset{h \to 0}{lt}\ h \sum_{m=0}^{n-1} f(mh),$

where $nh = 1$

$$= \underset{h \to 0}{lt}\ h \Big[\beta + [\alpha h^2 + \beta] + [\alpha (2h)^2 + \beta] + \ldots\ldots + \alpha[(n-1)h]^2 + \beta\Big]$$

$$= \underset{h \to 0}{lt}\ h \Big[n\beta + \alpha h^2\ (1^2 + 2^2 + 3^2 + \ldots\ \ldots\ldots + (n-1)^2\Big]$$

$$= \underset{h \to 0}{lt}\ h \left[n\beta + \alpha h^2 . \frac{n(n-1)(2n-1)}{6}\right]$$

$$= \underset{h \to 0}{lt} \left[(nh)\beta + \frac{\alpha}{6}\, nh(nh - h)(2nh - h)\right]$$

$$= \underset{h \to 0}{lt} \left[1 \times \beta + \frac{\alpha}{6} \times 1\,(1 - h)(2 - h)\right]$$

$$= \beta + \frac{\alpha}{3} \quad \text{(Ans.)}$$

6.3 Fundamental theorem of integral calculus

Now, we shall give the statement of the fundamental theorem of integral calculus. This theorem actually correlates the two apparently dissimilar definitions of integration. From the statement of the theorem it is evident that the two definitions of integration are identical.

If $f(x)$ is integrable in $(a\ \ b)$, $b > a$ and if there exists a function $\phi(x)$ such that $\phi'(x) = f(x)$ in $(a\ b)$ then $\displaystyle\int_a^b f(x)\, dx = \phi(b) - \phi(a)$.

We have seen in the last article that integral of $f(x)$ with respect to x within limits a to b, i.e. $\displaystyle\int_a^b f(x)\, dx = \underset{h \to 0}{lt}\ h \sum_{m=0}^{n-1} f(a + mh).$ With this definition in mind, we find that the above theorem establishes a connection between integration as a particular kind of summation and integration as the inverse operation of differentiation. Thus the two definitions of integration are identical.

6.4 Working rule for evaluation of definite integral

The fundamental theorem of integral calculus makes the problem of evaluating a definite integral much easier. If we are to integrate a function $f(x)$ within limits a to b, we should first of all determine the value of the indefinite integral $\int f(x)dx$ by the usual methods.

Suppose $\int f(x)\, dx = \phi(x) + C$ where C is the arbitrary constant of integration. Then $\displaystyle\int_a^b f(x)dx = \Big[\phi(x) + C\Big]_a^b =$ value of $\{\phi(x) + C\}$ for $x = b$ minus the value of $\{\phi(x) + C\}$ for $x = a = \phi(b) + C - \phi(a) - C = \phi(b) - \phi(a)$. We note that in definite integral the arbitrary constant of integration does not appear.

6.5 Change of limits of a definite integral

We have seen that if we are to evaluate a definite integral we must determine the value of the corresponding indefinite integral first. While evaluating an indefinite integral we may have to use the method of substitution. Very often it becomes rather inconvenient to transform the result back into the original variable and then substitute the limits. This is avoided by changing the limits of the given integral suitably corresponding to the change in the variable.

6.6 Illustrative examples

EXAMPLE 1

Integrate $\displaystyle\int_0^a \sqrt{a^2 - x^2}\, dx.$

Put $x = a \sin \phi \qquad \therefore\quad dx = a \cos \phi\ d\phi$

Again, when $x = 0$, $\phi = 0$ and when $x = a$, $\phi = \frac{\pi}{2}$.

So, $\int_0^a \sqrt{a^2 - x^2}\, dx = a^2 \int_0^{\frac{\pi}{2}} \cos^2\phi \; d\phi$

$$= \frac{a^2}{2} \int_0^{\frac{\pi}{2}} [1 + \cos 2\phi] \; d\phi$$

$$= \frac{a^2}{2}\left[\phi + \frac{\sin 2\phi}{2}\right]_0^{\frac{\pi}{2}} = \frac{a^2}{2}\left(\frac{\pi}{2}\right) = \frac{\pi a^2}{4}$$

EXAMPLE 2

Integrate $\int_0^a \frac{e^{m \tan^{-1} x}}{(1 + x^2)}\, dx$

Put $\tan^{-1}x = z$ then $\frac{dz}{dx} = \frac{1}{1+x^2}$

Again when $x = 0$, $z = 0$ and when $x = a$, $z = \tan^{-1}(a)$

So the integral becomes

$$\int_0^{\tan^{-1} a} e^{mz}\, dz = \left[\frac{e^{mz}}{m}\right]_0^{\tan^{-1} a}$$

$$= \frac{e^{m \tan^{-1} a}}{m} - \frac{1}{m} = \frac{1}{m}\left\{e^{m \tan^{-1} a} - 1\right\}$$

EXAMPLE 3

Show that

$$\int_a^b \frac{\log x \;\; dx}{x} = \tfrac{1}{2} \log \left(\frac{b}{a}\right) \log (ab)$$

Let us first evaluate $\int \frac{\log x\, dx}{x}$ by parts. Then

$$\int \frac{\log x\, dx}{x} = \log x \int \frac{dx}{x} - \int \left(\int \frac{dx}{x}\right) \frac{1}{x}\, dx$$

$$= (\log x)^2 - \int \frac{\log x\, dx}{x}$$

So, $2 \int \frac{\log x\, dx}{x} = (\log x)^2$

Or, $\int \frac{\log x \; dx}{x} = \frac{1}{2}(\log x)^2$

So $\int_a^b \frac{\log x\, dx}{x} = \left[\; \frac{1}{2}(\log x)^2 \right]_a^b = \frac{1}{2} \log \left(\frac{b}{a}\right) \log (ab)$

Exercise 6.1

1. Evaluate from the first principles the values of the following integrals:

(i) $\int_a^b e^x dx$ (ii) $\int_a^b e^{-kx}\, dx$

(iii) $\int_0^1 x^2\, dx$ (iv) $\int_0^1 (ax^3 + b)\, dx$

2. Evaluate the following integrals:

(i) $\int_0^1 \frac{dx}{x^2 + 1}$ (ii) $\int_0^{2a} \sqrt{2ax - x^2}\, dx$

(iii) $\int_0^{\frac{\pi}{2}} \sin x \sin 2x\, dx$ (iv) $\int_1^{\sqrt{e}} x \log x\, dx$

(v) Show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin x\, dx}{\sin x + \cos x} = \frac{\pi}{4}$$

(vi) $\int_{-c}^{+c} \frac{x e^{x^2}}{1 + x^2}\, dx = 0$

(vii) $\int_0^{\frac{\pi}{2}} x^3 \sin 3x \, dx = \frac{2}{27} - \frac{\pi^2}{12}$

(viii) $\int_0^{\frac{3\pi}{4}} \frac{\sin\theta \, d\theta}{1 + \cos^2\theta} = \frac{\pi}{4} + \tan^{-1}\left[\frac{1}{\sqrt{2}}\right]$

(ix) $\int_0^{\frac{\pi}{2}} \sin^6\phi \ \cos^3\phi \, d\phi = \frac{2}{63}$

(x) $\int_0^1 \frac{x\,dx}{\sqrt{1 + x^2}} = \sqrt{2} - 1$

3. Find the values of:

(i) $\int_0^1 \sqrt{\frac{1+x}{1-x}} \, dx$ (ii) $\int_1^2 \frac{dx}{e^x+1}$

(iii) $\int_0^a \frac{dx}{\sqrt{2ax - x^2}}$ (iv) $\int_0^1 \frac{(2x + 3)}{\sqrt{x^2 + x + 1}} \, dx$

(v) $\int_3^4 \frac{dx}{\sqrt{x^2 - 7x + 12}}$

7 Application of Integral Calculus (Quadrature)

7.1 Areas of plane curves

We have seen before that the area bounded by any curve $y = f(x)$, the two ordinates $x = a$, $x = b$ and the x-axis is given by the definite integral $\int_a^b f(x)\,dx$ or $\int_a^b y\,dx$. Similarly the area bounded by some curve $x = \phi(y)$ and the two abscissae (say) $y = c$, $y = d$ and the y-axis is given by the integral

$$\int_c^d x\,dy = \int_c^d \phi(y)\,dy$$

7.2 Illustrative examples

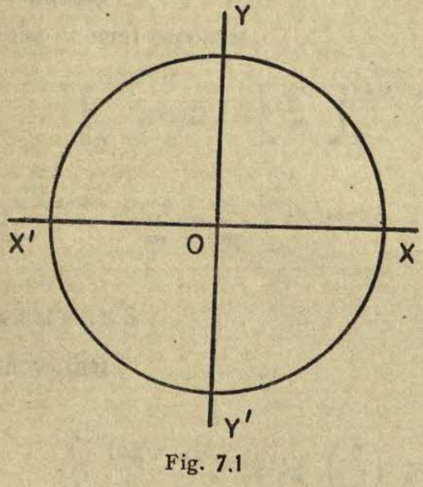


Fig. 7.1

EXAMPLE 1

Find the area of a circle whose equation is $x^2 + y^2 = a^2$.

The radius of the circle is a and the centre o is the origin. The area of the circle = $4 \times$ Area of a quadrant $= 4\int_0^a \sqrt{a^2 - x^2}\,dx$.

Put $x = a\sin\theta$ then $dx = a\cos\theta\,d\theta$.

So the area

$$= 4\int_0^{\frac{\pi}{2}} a^2\cos^2\theta\,d\theta = 4a^2\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= 2a^2\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)\,d\theta$$

$$= 2a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = 2a^2\left\{\frac{\pi}{2}\right\} = \pi a^2.$$

EXAMPLE 2

Find the area of a hyperbola $xy = c^2$ bounded by the x-axis and the ordinates $x = a$ and $x = b$.

$$\text{Area} = \int_a^b \frac{c^2}{x}\,dx = c^2\Big[\log x\Big]_a^b$$

$$= c^2\left\{\log b - \log a\right\} = c^2\log\left(\frac{b}{a}\right)$$

EXAMPLE 3

Find the area of the segment cut off from $y^2 = 4x$ by the line $y = 2x$.

$y^2 = 4x$ is a parabola whose vertex is the origin. $y = 2x$ is a straight line passing through the origin. Let these two meet at the point A. Then the coordinates of the point A can be determined by solving these two equations.

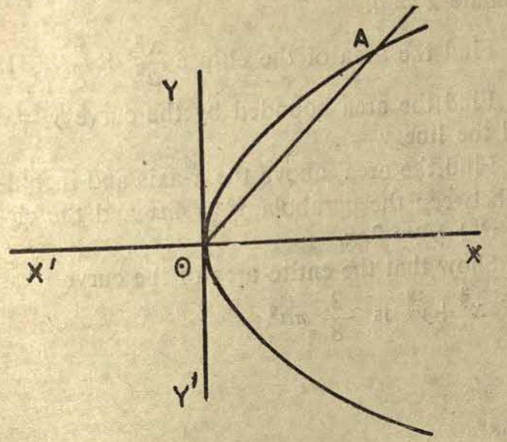


Fig. 7.2

Eliminating y out of these two equations we find,

$$x^2 - x = 0$$

Or $$x(x-1) = 0$$

Hence $$x = 0 \text{ or } x = 1$$

$x = 0$ corresponds the origin 0. So the x-coordinate of the point of intersection of the parabola and the straight line is 1. The area bounded by the parabola and the given straight line = Area bounded by the parabola, the ordinate $x = 1$ and the x-axis—Area bounded by the straight line $y = 2x$, the ordinate $x = 1$ and the x-axis.

Hence the required area

$$= \int_0^1 2\sqrt{x}\, dx - \int_0^1 2x\, dx$$

$$= \frac{4}{3}\left(x^{\frac{3}{2}}\right)_0^1 - \left(x^2\right)_0^1 = \tfrac{4}{3} - 1 = \tfrac{1}{3}.$$

EXAMPLE 4

Find the area bounded by the curves $y^2 = 4ax$ and $x^2 = 4ay$.

$y^2 = 4ax$ and $x^2 = 4ay$ are the two parabolas. Let us suppose they intersect at the point A. The x-coordinate of the point of intersection can be obtained eliminating y from the two given equations. Eliminating y we find

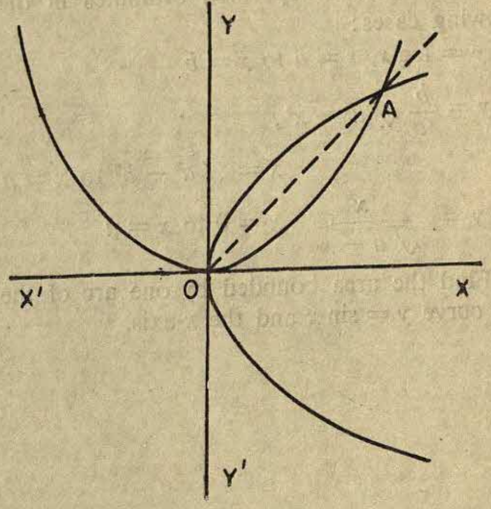


Fig. 7.3

$$4ax = \left(\frac{x^2}{4a}\right)^2 = \frac{x^4}{16a^2}$$

or $x^4 = 64a^3x$

or $x(x^3 - 64a^3) = 0$

So either $x = 0$ or $x^3 = 64a^3$ or $x = 4a$.

$x = 0$, corresponds the origin. Hence the x-coordinate of the point A is $4a$. Therefore the required area

$$= \int_0^{4a} 2\sqrt{a}\sqrt{x}\, dx - \int_0^{4a} \frac{x^2}{4a}\, dx$$

$$= \frac{4}{3}\sqrt{a}\left(x^{\frac{3}{2}}\right)_0^{4a} - \frac{1}{4a}\left(\frac{x^3}{3}\right)_0^{4a}$$

$$= \frac{32}{3}a^2 - \frac{32a^2}{6} = \frac{16a^2}{3}$$

Exercise 7.1

1. Obtain the areas bounded by the curve, the x-axis and the specified ordinates in the following cases:

(i) $y = \log x$, $x = a$ to $x = b$

(ii) $y = \frac{b}{a}\sqrt{a^2 - x^2}$,

$x = \sqrt{a^2 - b^2}$ to $x = a$

(iii) $y = \dfrac{x}{\sqrt{a - x}}$ $x = 0$ to $x = a$

2. Find the area bounded by one arc of the sine curve $y = \sin x$ and the x-axis.

3. Find by integration the area of the triangle bounded by the line $y = 9x$, the x-axis and the ordinate $x = 4$.

4. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

5. Find the area bounded by the curve $y^2 = x^3$ and the line $y = x$.

6. Find the area, above the x-axis and included between the parabola $y^2 = 4ax$ and the circle $x^2 + y^2 = 2ax$.

7. Show that the entire area of the curve

$x^{\frac{2}{3}} + y^{\frac{2}{3}}$ is $\frac{3}{8}\pi a^2$

DIFFERENTIAL EQUATIONS

8 First Order Differential Equations

8.1 Introduction and classification

We have already studied ordinary algebraic equations which may be linear or non-linear (quadratic) in nature. Solution of simultaneous equations in several variables is also well known to us. Here we shall discuss the solution of another kind of equation which involves the variables as well as their derivatives called **differential equations.** These equations are found to occur in all branches of science and technology and it has been observed that the ultimate solution of almost all physical problems rests on the solution of some differential equation. Let us consider a few examples before we proceed to give the formal definition.

$$\text{(i)}\quad (4x+3)\frac{dy}{dx}+(7x^2+8y)=0$$

$$\text{(ii)}\quad \frac{dy}{dx}=y+\frac{1}{y}\qquad \text{(iii)}\quad ydx-xdy=xydx$$

$$\text{(iv)}\quad x\frac{dy}{dx}=1+\left(\frac{dy}{dx}\right)^2$$

$$\text{(v)}\quad \frac{d^2y}{dx^2}+5\frac{dy}{dx}+4y=\sin x$$

$$\text{(vi)}\quad x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+n^2y=0$$

$$\text{(vii)}\quad \left(\frac{d^2y}{dx^2}\right)^2-3\frac{dy}{dx}+2y=x^2$$

These are all examples of differential equations. Careful consideration of the above equations suggests the following definition:

A differential equation is an equation that involves the differentials or the differential coefficient of the dependent variable with regard to the independent variable and the variable themselves.

Again, if we compare the first four differential equations of the above set with the last three of the same set we find that the first four involve the first derivatives only whereas the last three contain second derivatives. Therefore (i), (ii), (iii), (iv) are called **first order differential equations** and (v), (vi), (vii) are called **second order differential equations.**

Thus we find that the order of a differential equation is the order of the highest derivative occurring in it. Now let us see how to distinguish linear (first degree) and non-linear differential equations. If in a differential equation, the degree of the differential coefficient is one then it is called a **linear or first degree differential equation.** Examples (i), (ii), (iii), (v), (vi) are linear differential equations. If the degree of the differential coefficient is more than one then the corresponding differential equation is called non-linear (iv), (vii), etc. are examples of **non-linear differential equations.**

All these equations discussed above are **ordinary differential equations** as they involve only two variables one dependent and the other independent.

REMARKS

A differential equation is called **partial** if there are two or more independent variables and it involves partial derivatives of the dependent

dent variable with respect to the independent variables.

In this book we shall restrict ourselves only to the solution of ordinary first and second order linear differential equations.

8.2 Formation and solution of differential equations

Let us consider the relation $y = e^{ax}$. Differentiating both sides with respect to x we find $\frac{dy}{dx} = ae^{ax}$. Now eliminating e^{ax} from these two we get $\frac{dy}{dx} = ay$. This is a **first order differential equation.** $y = e^{ax}$ is called the **primitive of the differential equation.** Again, since $y = e^{ax}$ satisfies the first order differential equation.

$$\frac{dy}{dx} - ay = 0$$

It is a solution of the equation. If we substitute $y = Ce^{ax}$ where C is any constant, in the differential equation $\frac{dy}{dx} - ay = 0$ it is also satisfied. This shows that $y = Ce^{ax}$ is also a solution. Now writing for C any particular value 2, $-\frac{1}{3}, \frac{5}{7}$, etc. we get a large number of relations all of which satisfy the differential equation $\frac{dy}{dx} - ay = 0$. This shows that a differential equation can have an infinite number of solutions. These solutions are called **particular solutions of the differential equation** as these are obtained by putting particular values to the arbitrary constant C. Thus the general solution of the first order differential equation is $y = Ce^{ax}$.

Let us again consider the relation $z = At + B$ where A, B are any two arbitrary constants. Differentiating the above relation twice with respect to t we get the second order differential equation.

$$\frac{d^2z}{dt^2}=0$$

Now putting any particular value whatsoever for the two arbitrary constants A and B we get infinite number of relations, all of which satisfy the differential equation $\frac{d^2z}{dt^2} = 0$. These are the particular solutions of the equation. The **general solution of the differential equation** $\frac{d^2z}{dt^2} = 0$ is $z = At + B$, which contains two arbitrary constants. This can be extended to a differential equation of ηth order and it can be shown that the general solution of ηth order differential equation contains η arbitrary constants.

8.3 Geometrical significance of the differential equations

We have seen that a differential equation can have an infinite number of solutions. If we plot all these solutions, we shall get infinite number of curves having some common property as is evident from their graphs. Since these solutions are obtained from the same differential equation, the common characteristic is represented by the differential equation itself. This is the **geometrical significance of the differential equation.**

8.4 Solution of the first order differential equations by the method of separation of variables

Any first order differential equation can be written in the form

$$fdx + \phi dy = 0$$

where f and ϕ are functions of the two variables x and y or constants. If this differential equation can be written in the form $Pdx + Qdy = 0$, where P and Q are functions of x and y respectively then it can be integrated easily and the general solution of the differential equation

$$f(x, y)\,dx + \phi(x, y)\,dy = 0$$

can be obtained. Hence integrating, the general solution to the differential equation is given by

$$P_1(x) + Q_1(y) = C$$

where $P_1(x) = \int P(x)\,dx$, $Q_1(y) = \int Q(y)\,dy$

and C is the arbitrary constant. This method of solving the given differential equation $fdx + \phi dy = 0$ after rewriting the same in the form $P(x)\,dx + Q(y)\,dy = 0$ is called **solution by the method of separation of variables.**

8.5 Illustrative examples

EXAMPLE 1

Solve $x^2\dfrac{dy}{dx} + y = 1$

Separating the two variables, the above differential equation can be written as

$$\frac{dx}{x^2} + \frac{dy}{y-1} = 0$$

Now, integrating, we find

$$-\frac{1}{x} + \log(y-1) = \log a$$

where the arbitrary constant C is written in the form $\log a$, a being a constant.

So, $\dfrac{1}{x} = \log\left\{\dfrac{y-1}{a}\right\}$

Or, $y - 1 = ae^{\frac{1}{x}}$

EXAMPLE 2

Solve $\dfrac{dy}{dx} + 1 = e^{x-y}$

To solve this differential equation by the method of separation of variables, let us put $x - y = z$. Using the above transformation the differential equation becomes

$$\frac{dz}{dx} = 2 - e^z$$

$$\text{Or, } dx = \frac{dz}{2 - e^z}$$

Now, the general solution can be obtained by integrating both sides and adding the arbitrary constant C.

So we find

$$x + C = \int \frac{dz}{2 - e^z}$$

To integrate the right hand side let us put

$$2 - e^z = u$$

Now, $\displaystyle\int \frac{dz}{2 - e^z} = \int \frac{du}{u(u-2)}$

$$= \frac{1}{2}\int\left\{\frac{1}{u-2} - \frac{1}{u}\right\}du$$

$$= \frac{1}{2}\Big[\log(u-2) - \log u\Big] = \tfrac{1}{2}\log\left(\frac{u-2}{u}\right)$$

$$= \frac{1}{2}\log\left(\frac{e^z}{e^z - 2}\right)$$

Reverting to the original variables we get the general solution of the differential equation as

$$x + C = \tfrac{1}{2}\log\left\{\frac{e^{x-y}}{e^{x-y} - 2}\right\}$$

EXAMPLE 3

Solve $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$

Separating the two variables, the given differential equation can be written in the form

$$\frac{dx}{x+a} = \frac{dy}{y(1-ay)}$$

Now integrating we find

$$\log(x+a) + \log C = \int \frac{dy}{y(1-ay)}$$

Now, $\displaystyle\int \frac{dy}{y(1-ay)} = \int\left\{\frac{1}{y} + \frac{a}{1-ay}\right\}dy$

$$= \log y - \log(1 - ay)$$

So, $\log(x+a) + \log C = \log y - \log(1 - ay)$

Or, $\log C(x+a) = \log\left[\dfrac{y}{1-ay}\right]$

Or $C(x+a) = \dfrac{y}{1-ay}$

So the general solution of the differential equation is

$$y = C(x+a)(1-ay)$$

EXAMPLE 4

Solve $x^2dy + y(x+y)\,dx = 0$

This differential equation can be written in the form $\dfrac{dy}{dx} = \dfrac{-y(x+y)}{x^2}$.

Here we find that the numerator and the denominator of the term on the right hand side are homogeneous in x and y of degree 2. Now let us put $y = vx$. With this substitution the differential equation becomes

$$\frac{dv}{v^2 + 2v} = -\frac{dx}{x}$$

Or, $\frac{1}{2}\left[\frac{1}{v} - \frac{1}{v+2}\right] dv = -\frac{dx}{x}$

Integrating both sides we get the solution as

$\frac{1}{2}\log\frac{v}{v+2} = -\log x + \log c$ (where $\log c$ is the integration constant).

Or, $\log\sqrt{\frac{y}{y+2x}} = \log\left(\frac{c}{x}\right)$

Or, $\sqrt{\frac{y}{y+2x}} = \frac{c}{x}$

Or, $x\sqrt{y} = c\,(y + 2x)^{\frac{1}{2}}$

EXAMPLE 5

Solve $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

This differential equation can be written as

$$\frac{dy}{dx} = \frac{y^2}{x^2} - \frac{y}{x} = \frac{y^2x - yx^2}{x^3}$$

Here too we find that the numerator and the denominator of the term on the right hand side of the given differential equation are homogeneous functions in x and y of degree 3. So let us put $y = vx$. Differentiating with respect to x we find

$$\frac{dy}{dx} = v + \frac{x\,dv}{dx}$$

So the given differential equation becomes

$$v + \frac{x\,dv}{dx} = \frac{v^2x^3 - vx^3}{x^3}$$

$$= v^2 - v$$

Or, $x\frac{dv}{dx} = v^2 - 2v$

Or, $\frac{dv}{v^2 - 2v} = \frac{dx}{x}$

Or, $\frac{1}{2}\left[\frac{1}{v-2} - \frac{1}{v}\right] dv = \frac{dx}{x}$

Now integrating both sides and replacing v by $\frac{y}{x}$ and adding the arbitrary constant C we find the general solution as

$$\log\left(\frac{y-2x}{y}\right)^{1} = \log x + C$$

Or, $\log\left[\frac{\sqrt{y-2x}}{x\sqrt{y}}\right] = C$

Or, $\frac{\sqrt{y-2x}}{x\sqrt{y}} = e^{0}$

$e^{c} = d$ (say) (another constant)

$\therefore$ The general solution

$$dx\sqrt{y} = \sqrt{y - 2x}$$

REMARK

This type of differential equations are called **homogeneous differential equations.** To solve these equations we must substitute $y = vx$ first and then separate the variables.

EXAMPLE 6

Solve $(2x - 2y + 5)\,dy$
$- (x - y + 3)\,dx = 0$

This equation can be written in the form

$$\frac{dy}{dx} = \frac{x - y + 3}{2x - 2y + 5}$$

To solve this differential equation, let us put

$$x - y + 3 = z$$

Differentiating with respect to x we find

$$\frac{dz}{dx} = 1 - \frac{dy}{dx}$$

So the given differential equation becomes

$$1 - \frac{dz}{dx} = \frac{z}{2z-1}$$

Or, $\frac{dz}{dx} = \frac{z-1}{2z-1}$

Or, $\frac{(2z-1)}{z-1}\,dz = dx$

$$\frac{(2z-2+1)}{z-1}\,dz = dx$$

Now, integrating both sides we find

$$\int 2dz + \int\frac{dz}{z-1} = x + C$$

Or $2z + \log(z - 1) = x + C$.

Now replacing z by $x - y + 3$ we find

$$2(x - y + 3) + \log\ (x - y + 2) = x + C$$

EXAMPLE 7

Solve $\dfrac{dy}{dx} = \dfrac{6x - 2y - 7}{2x + 3y - 6}$

To solve this differential equation let us introduce the new variables ξ, η defined by the relations

$$x = \xi + h \text{ and } y = \eta + k$$

where h and k are constants.

The differential equation in terms of the new variables becomes

$$\frac{d\eta}{d\xi} = \frac{6\xi - 2\eta + 6h - 2k - 7}{2\xi + 3\eta + 2h + 3k - 6}$$

Let us choose the constants h and k in such a way that $6h - 2k - 7 = 0$ and $2h + 3k - 6 = 0$. Solving these two equations, we have $h = \frac{3}{2}$, $k = 1$.

So the equation becomes

$$\frac{d\eta}{d\xi} = \frac{6\xi - 2\eta}{2\xi + 3\eta}$$

This is a homogeneous differential equation. To solve this equation let us put $\eta = v\xi$.

Differentiating with respect to ξ we find

$$\frac{d\eta}{d\xi} = v + \xi\ \frac{dv}{d\xi}$$

Putting these values in the above differential equation we find

$$\frac{d\xi}{\xi} = -\frac{1}{2}\ \frac{(6v + 4)\,dv}{[3v^2 + 4v - 6]}$$

This on integration gives

$$\log a + \log \xi = -\tfrac{1}{2} \log (3v^2 + 4v - 6)$$

where $\log a$ is the integration constant.

Or, $\log a\xi = \log (3v^2 + 4v - 6)^{-\frac{1}{2}}$

Or $a\xi = (3v^2 + 4v - 6)^{-\frac{1}{2}}$

Now putting the values of ξ and v in term of x and y we get the solution as

$$a\left(x - \tfrac{3}{2}\right) = \left[\ 3\left[\frac{2(y-1)}{2x-3}\right]^2 + 4\left[\frac{2(y-1)}{2x-3}\right] - 6\ \right]^{-\frac{1}{2}}$$

Exercise 8.1

1. Solve the following differential equations:

(i) $y(1 + x)\,dx + x(1 + y)\,dy = 0$

(ii) $(x + y)^2 \dfrac{dy}{dx} = a^2$

(iii) $\dfrac{dy}{dx} = x^2 e^y + e^{x+y}$

(iv) $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$

(v) $\sin^{-1}\left(\dfrac{dy}{dx}\right) = x + y$

(vi) $\dfrac{dy}{dx} = \dfrac{3x + 2y}{2x - 3y}$

(vii) $(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy = 0$

(viii) $\dfrac{1}{2x}\dfrac{dy}{dx} + \dfrac{x + y}{x^2 + y^2} = 0$

(ix) $x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$

(x) $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\dfrac{y}{x}$

(xi) $\dfrac{dy}{dx} = \dfrac{\alpha x + \beta y - \alpha}{\beta x + \alpha y - \beta}$

(xii) $(2x + 3y - 8)\,dx = (x + y - 3)dy$

(xiii) $\dfrac{dy}{dx} = \dfrac{x + y - 3}{1 - x + y}$

(xiv) $\dfrac{dy}{dx} = \dfrac{x + y}{x + y - 1}$

(xv) $(2x + 3y - 5)\dfrac{dy}{dx} + (3x + 2y - 5) = 0$

REMARK

Solution of first order differential equations by various other methods such as multiplying by the integrating factor and exact equations, etc. have not been discussed here.

9 Second Order Differential Equations

THE MOST general second order differential equation is of the form

$$\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = X(x) \qquad (1)$$

where P_1, P_2 and X are either functions of the independent variable x or are constants. Here, we shall restrict ourselves only to second order differential equations with constant coefficients (P_1, P_2, etc. are constants) and when there is no term on the right hand side.

9.1 Solution of second order differential equations with constant coefficients

Let us start with the differential equation of the form

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2y = 0 \qquad (2)$$

where P_1, P_2 are constants. Obviously we have taken $X = 0$. Let us assume $y = e^{mx}$ to be a solution of equation (2). Then this should satisfy it. Substituting in the differential equation we find

$$(m^2 + P_1m + P_2)\, e^{mx} = 0$$

As $e^{mx} \neq 0$, $m^2 + P_1m + P_2 = 0$ (3)

This equation is called the **auxiliary equation.** Since it is a quadratic equation in m it should have two roots m_1 and m_2 (say). Then $y = e^{m_1x}$ and $y = e^{m_2x}$ are two particular solutions of the second order differential equation. It can be seen easily that $y = c_1e^{m_1x}$ and $y = c_2e^{m_2}$, where c_1 and c_2 are any two arbitrary constants, are also solutions. As the general solution of a second order differential equation should contain two arbitrary constants $y = c_1e^{m_1x} + e_2c^{m_2x}$ is taken as the general solution of the second order differential equation (2), when the two roots m_1 and m_2 of the quadratic equation in m are real and distinctly different.

If the two roots of the auxiliary equation are equal, i.e. $m_1 = m_2$, then the general solution of the differential equation cannot be obtained by the above rule. Since $m_1 = m_2$, the solution becomes

$y = c_1e^{m_1x} + c_2e^{m_1x} = (c_1 + c_2)\, e^{m_1x} = ce^{m_1}$

Thus we see that the solution $y = ce^{m_1x}$ is no longer the general solution as it involves only one arbitrary constant.

Now let us work out the general solution when the two roots of auxiliary equations are equal.

Since the roots are equal we can take $m_1 = m_2 = \alpha$. Then the sum and the product of the roots become $m_1 + m_2 = 2\alpha$ and $m_1m_2 = \alpha^2$. So we can rewrite the second order differential equation (2) in the form

$$\frac{d^2y}{dx^2} - 2\alpha\frac{dy}{dx} + \alpha^2y = 0 \qquad (4)$$

Let us assume $y = e^{\alpha x}v(x)$ be a trial solution of equation (4). Putting $y = e^{\alpha x}v(x)$ in equation (4) we find

$$\frac{d^2v}{dx^2}\, e^{\alpha x} = 0$$

As $e^{\alpha x} \neq 0$, $\dfrac{d^2v}{dx^2} = 0$

Its solution is given by

$$v(x) = c_1x + c_2$$

where c_1 and c_2 are two arbitrary constants. Therefore the general solution of the second order differential equation (2) when the two roots of the auxiliary equations are equal is given by

$$y = (c_1x + c_2)\, e^{\alpha x}$$

Lastly, to determine the general solution of the second order differential equation when the roots of the auxiliary equation are not real we proceed as follows.

If the two roots m_1 and m_2 are not real, they will be complex in nature. Moreover, we know that complex roots occur in pairs. So if $m_1 = \alpha + i\beta$ then $m_2 = \alpha - i\beta$. α and β are real and $i = \sqrt{-1}$. In this case the general solution of the differential equation (2) is

$$\begin{aligned} y &= c_1 e^{(\alpha+i\beta)x} + c_2\, e^{(\alpha-i\beta)x} \\ &= e^{\alpha x}\left\{ c_1 e^{i\beta x} + c_2\, e^{-i\beta x} \right\} \\ &= e^{\alpha x}\left\{ c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right\} \\ &= e^{\alpha x}\left\{ (c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x\right\} \end{aligned}$$

which we can write as

$$y = e^{\alpha x}\,[A\cos\beta x + B\sin\beta x].$$

where $A = c_1 + c_2$ and $B = i(c_1 - c_2)$ are the two arbitrary constants.

9.2 Illustrative examples

EXAMPLE 1

Solve $\dfrac{d^2y}{dx^2} + 5\,\dfrac{dy}{dx} + 4y = 0$

Let $y = e^{mx}$ be a solution of the above equation.

Then $(m^2 + 5m + 4)\, e^{mx} = 0$

As $e^{mx} \neq 0$, $m^2 + 5m + 4 = 0$

The two roots of this equation are -1 and -4.

$\therefore$ the general solution is $y = c_1e^{-x} + c_2e^{-4x}$.

EXAMPLE 2

Solve $\dfrac{d^2y}{dx^2} + 2\,\dfrac{dy}{dx} + y = 0$

Let $y = e^{mx}$ be a solution of the equation.

Then $(m^2 + 2m + 1)\, e^{mx} = 0$

As $e^{mx} \neq 0$, $m^2 + 2m + 1 = 0$

The two roots of the auxiliary equation $m^2 + 2m + 1 = 0$ are equal. The roots are $-1\ -1$. So the general solution is

$$y = (c_1x + c_2)\, e^{-x}$$

EXAMPLE 3

Solve $\dfrac{d^2y}{dx^2} - 4\,\dfrac{dy}{dx} + 13y = 0.$

Let $y = e^{mx}$ be a trial solution of the equation.

Then $(m^2 - 4m + 13)\, e^{mx} = 0$

As $e^{mx} \neq 0$, $m^2 - 4m + 13 = 0$

Here, the auxiliary equation $m^2 - 4m + 13 = 0$ has got imaginary roots. The roots are $2 \pm i3$. So the general solution is

$$y = e^{2x}\,\{A\cos 3x + B\sin 3x\}$$

where A and B are arbitrary constants.

EXAMPLE 4

Solve $\dfrac{d_2y}{dx^2} + y = 0$ when $x = 0$,

$$y = 4;\ \text{when } x = \frac{\pi}{2},\ y = 0$$

Let $y = e^{mx}$ be a trial solution of the equation. Then

$$e^{mx}(m^2 + 1) = 0.$$

As $e^{mx} \neq 0$, $m^2 + 1 = 0$

$$\therefore\ m = \pm i$$

So the general solution is

$$y = A\cos x + B\sin x$$

where A and B are arbitrary constants.

When $x = 0$, $y = 4$

So $A = 4$.

Again, when $x = \dfrac{\pi}{2}$, $y = 0$

So $B = 0$.

Thus we find that $A = 4$ and $B = 0$. Putting these values in the general solution $y = A\cos x + B\sin x$ we get

$$y = 4\cos x$$

This is the solution of the differential equation under prescribed boundary conditions.

EXAMPLE 5

Show that if $l\,\dfrac{d^2\theta}{dt^2} + g\theta = 0$ and if $\theta = \alpha$ and $\dfrac{d\theta}{dt} = 0$, when $t = 0$ then

$$\theta = \alpha\cos\left\{ t\sqrt{\frac{g}{l}} \right\}$$

Let $\theta = e^{mt}$ be a trial solution. Then the auxiliary equation is $m^2 + \dfrac{g}{l} = 0$.

$$\therefore \quad m = \pm i\sqrt{\frac{g}{l}}$$

So the general solution of the differential equation

$$\theta = A\cos\sqrt{\frac{g}{l}}\,t + B\sin\sqrt{\frac{g}{l}}\,t$$

where A and B are the two arbitrary constants

Given, when $t = 0$, $\theta = \alpha$

So $\alpha = A$

Now differentiating the general solution with regard to t we find

$$\frac{d\theta}{dt} = -A\sqrt{\frac{g}{l}}\sin\sqrt{\frac{g}{l}}\,t + B\sqrt{\frac{g}{l}}\cos\sqrt{\frac{g}{l}}\,t$$

Again $\frac{d\theta}{dl} = 0$ at $t = 0$

So $0 = B\sqrt{\frac{g}{l}}$

As $\sqrt{\frac{g}{l}} \neq 0$, $B = 0$

Hence the solution under the given conditions is given by

$$\theta = \alpha\cos\left\{\sqrt{\frac{g}{l}}\,t\right\}$$

Exercise 9.1

1. Solve the following equations:

(i) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

(ii) $\frac{d^2y}{dx^2} + (a+b)\frac{dy}{dx} + aby = 0$

(iii) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

(iv) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$

(v) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

(vi) $\frac{d^2y}{dx^2} + cy = 0$

(vii) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

(viii) $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 2y = 0$

(ix) $\frac{d^2z}{dt^2} - 2\frac{dz}{dt} + 4z = 0$

(x) $\frac{d^2s}{dt^2} - 3\frac{ds}{dt} + s = 0$

2. Solve under the given conditions:

(i) $\frac{d^2y}{dz^2} + 4\frac{dy}{dz} - 12y = 0$; when $y = 0$ and $\frac{dy}{dz} = 1$ at $z = 0$

(ii) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$, when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 2$ at $x = 0$

(iii) $\frac{d^2x}{dt^2} + \mu^2x = 0$, when $t = 0$ and $\frac{dx}{dt} = 0$ at $x = a$

(iv) $\frac{d^2u}{dx^2} + 2\frac{du}{dx} + u = 0$, when $x = 0$, $u = 3$ and $x = 0$, $\frac{du}{dx} = 5$

Exercise 9.2

1. Solve the following differential equations:

(i) $(e^x + 1)y\,dy = (y + 1)e^x\,dx$

(ii) $9\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 4y = 0$

(iii) $(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0$

(iv) $6\frac{d^2y}{dx^2} + 17\frac{dy}{dx} + 12y = 0$

(v) $x\sin\left(\frac{y}{x}\right)dy = \left\{y\sin\left(\frac{y}{x}\right) - x\right\}dx$

(vi) $(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0$.

2. Solve the following differential equations under the prescribed conditions:

(i) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$, when $x = 0$, $y = 0$; and when $x = 1$, $y = 2$

(ii) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 25y = 0$, when $x = \dfrac{\pi}{2}$,

$y = 0$, when $x = 0, \dfrac{dy}{dx} = 4$

3. Solve $\dfrac{d^2u}{dt^2} - 2\lambda\dfrac{du}{dt} + (\lambda^2 + \mu^2)\,y = 0$

where λ, μ are constants.

4. Solve $\dfrac{dv}{dt} = \sqrt{v - t}$

5. Solve $\dfrac{d^2x}{dt^2} + k\dfrac{dx}{dt} + \mu x = 0$

MECHANICS

10 Preliminary Ideas

10.1 Basic concepts

It would be better to clarify at the beginning certain basic concepts in the study of mechanics, such as:

(i) **Matter:** Matter loosely speaking is something that occupies space. It cannot be so correctly defined as space in mathematical terms. We speak of matter as something that has certain attributes such as it offers resistance, it undergoes extension, etc. Matter exists in three stages—solid, liquid and gas. In mechanics, we are primarily concerned with solids which are termed as **bodies** or more specifically **rigid bodies.**

(ii) **Body:** A body is a finite portion of matter the quantity of which is the mass of the body.

(iii) **Particle:** A particle is an infinitesimal portion of matter. It possesses mass but has no dimension. It is actually a point-mass. For this reason, a particle is represented by a geometrical point in mechanics.

(iv) **Rigid Body:** A body is an agglomeration of an infinite number of particles. It is called 'rigid' when the distance between any two of its particles remains invariable under all circumstances. The idea of a rigid body is an ideal one. It does not occur in reality. However, for all practical purposes, we shall use the term 'rigid body' in its true sense.

(v) **Units:** Every physical or measurable quantity has some unit which is expressed in terms of three fundamental units—length, mass and time which are denoted respectively by L, M and T. For example, volume has a unit L^3 and velocity has a unit LT^{-1}.

We generally use two systems of units—the **C.G.S.** (or French or metric) system and the **F.P.S.** (or British) system. In the C.G.S. (centimetre-gram-second) system, the units for length, mass and time are respectively centimetre, gram and second. In the F.P.S. system, length, mass and time are respectively expressed in foot, pound and second.

(vi) **Scalars and Vectors:** Physical quantities such as mass, temperature, density, etc. are measured by numbers referred to some chosen unit. These numbers are called scalars.

Quantities like displacement, velocity, acceleration and force which require for their complete specification a direction as well as a scalar, are called vectors. A vector is represented geometrically by a directed line segment as shown in Fig. 10.1.

Fig. 10.1

Here the line segment AB, directed from A to B, represents the *vector* AB which is denoted by the symbol $\overrightarrow{AB}$. $\overrightarrow{AB}$ has the length or magnitude $|\overline{AB}|$ and the direction from A to B as indicated by the arrow.

(vii) **Mechanics:** *The subject which deals with the action of forces on bodies or particles is known as mechanics.* Action forces on a body may lead to either a state of rest or a state of motion of the body. The subject which deals with the state of rest of bodies under the action forces is called **Statics.** The subject dealing with the motion of bodies under action of forces is known as **Dynamics.**

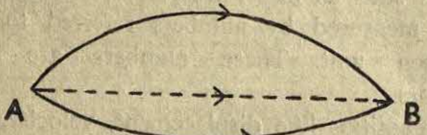
Dynamics is divided into two parts—Kinematics and Kinetics. **Kinematics** deals with the geometry of motion apart from the forces which produce it. On the other hand, **Kinetics** deals with the effect of forces on the motion of bodies.

We shall concentrate our attention on the motion of particles.

10.2 Some vector quantities used frequently in Mechanics

The physical quantities which are frequently used in mechanics are displacement, velocity, acceleration, force, tension and thrust. We define these terms as given below:

(i) **Displacement :** Let us suppose that a particle moves from position A to position B. The particle may move from A to B along any curved path joining A to B or along the straight line from A to B. Whatever path it may follow in moving from A to B, AB is called its displacement. The displacement is thus a vector quantity possessing (a) a magnitude—the length of the line segment AB, (b) a direction—along AB and (c) a sense—from A to B.



Fig; 10.2

(ii) **Velocity :** The rate of change of displacement of a particle as it moves from one position to another is known as its velocity.

(iii) **Acceleration :** The rate of change of velocity of a moving particle is called its acceleration.

In both (ii) and (iii), *the rate of change* is calculated with respect to time. We shall find out later expressions for velocity and acceleration of a moving particle when it moves along a straight line or on a plane.

(iv) **Force :** A force is that which changes or tends to change the state of rest or of uniform motion of a body (or a particle) on a straight line.

This is the qualitative definition of force. For practical purposes, the quantitative definition of the force is important. As we shall see later, both the qualitative and quantitative aspects of force are derived from Newton's laws of Motion.

(v) **Tension :** It is a kind of force that develops along a string when a body is pulled or supported by that string. It also goes by the name *pull.*

(vi) **Thrust :** When a body is pushed by a rod, a kind of force develops along the rod. This force is known as thrust or *push.*

DYNAMICS

11 Kinematics : Velocity

11.1 Speed and velocity

We shall consider a particle in motion. Let the particle describe distances s_1, s_2......at times t_1, t_2, ... The ratios $\frac{s_1}{t_1}$, $\frac{s_2}{t_2}$ ....are called the **average speed of the particle** at times t_1, t_2.........respectively.

When $\frac{s_1}{t_1} = \frac{s_2}{t_2} = \ldots. =$ (say) u, the particle is said to be moving with a **uniform speed** u. We measure the uniform speed by the distance described in unit time.

Speed is, therefore, the rate of describing distance.

As we have defined earlier, the velocity of a moving particle is the rate of change of its displacement. If d be its displacement in time t, $\frac{d}{t}$ is called the **average velocity.** The velocity is said to be uniform when $\frac{d}{t}$ is a constant whatever interval of time we may take for observation. *The velocity is, therefore, a vector quantity possessing a magnitude as well as a direction* (*the direction of its displacement*).

A displacement equal to unit distance described in unit time is the unit velocity. The unit of velocity is (i) cm. per sec. (cm./sec.) in the C.G.S. system and (ii) foot per sec. (ft./sec.) in the F.P.S. system.

The dimension of velocity is LT^{-1}.

When the particle moves along a straight line, its uniform speed and uniform velocity are the same. These two quantities are different when it moves along a curved path. For example, when the particle moves in a circle, its velocity is not uniform, although its speed (i.e. the rate of describing the circle) may be uniform because its velocity at any point on the circle being in the tangential direction there, continually changes direction.

Let a particle move from A to B in time t_1, B to C in time t_2 where $AB = s_1$, $BC = s_2$, $AC = d$ and $\angle ABC = 90°$.

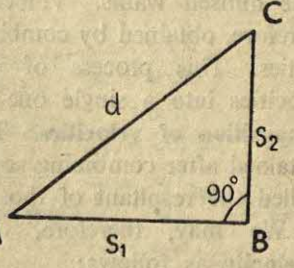


Fig. 11.1

Taking $t_1 + t_2 = t$, the speed of the particle at time $t = \frac{s_1+s_2}{t_1+t_2} = \frac{s_1+s_2}{t}$ and the velocity of the particle at time $t = \frac{d}{t} = \sqrt{\frac{s_1^2 + s_2^2}{t}}$

We should note that *speed is a scalar quantity* while *velocity is a vector quantity.*

11.2 Geometrical representation of velocity

Velocity being a vector quantity, it can be represented by a directed line segment. Let a point P (i.e. a particle at P) move with a velocity of 35 ft./sec. along line *PQ* in a direction from *P* to *Q*. Taking a line segment of length 1 inch to represent 7 ft./sec., we cut off the portion *PA* of length 5 inches from *PQ*. Then the line segment *PA*, directed from *P* to *A*, represents the velocity of 35 ft./sec. in both magnitude and direction. We may say that $\overrightarrow{PA}$ represents the velocity in this case.

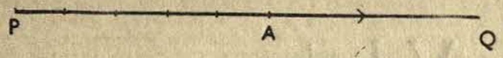


Fig. 11.2

11.3 Composition and resolution of velocities

It may happen that a moving point simultaneously possesses velocities in different directions. For example, when a man walks on the floor of a running bus, his velocity depends on (i) the velocity of the bus and (ii) the rate at which he himself walks. Velocity of the man is, therefore, obtained by combining these two velocities. This process of combining several velocities into a single one is known as the **composition of velocities**. The single velocity obtained after combining several velocities is called the resultant of those separate velocities. We may, therefore, define the **resultant velocity** as follows:

> If a moving point possesses, at the same time, two or more velocities v_1, v_2, …… and if a single velocity u can be found whose effect is the same as that of the velocities v_1, v_2……, this single velocity u is called the resultant of the other velocities and v_1, v_2 …… are called the components of u.

When a point possesses simultaneously two velocities in different directions, their resultant may be obtained by means of the following law:

Parallelogram of velocities: If two velocities of a moving point be represented in magnitude and direction by the two sides of a parallelogram drawn from one of its angular points, their resultant is represented both in magnitude and direction by the diagonal of the parallelogram passing through that angular point.

Formal proof of this fundamental rule of composition of velocities is not necessary for our purpose. We shall use it as a working principle. Graphical illustration of this principle is given below:

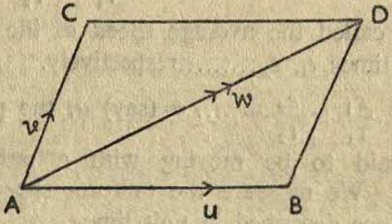


Fig. 11.3

Let a point *A* possess simultaneously two velocities *u* and *v* which are represented, in both magnitude and direction, by the two adjacent sides *AB* and *AC* respectively of the parallelogram *ABDC*.

Then the diagonal *AD* represents, in both magnitude and direction, the resultant velocity *w* of *u* and *v*. Here *u* and *v* are components of *w*. The process of deriving the component velocities *u* and *v* by breaking up the velocity *w* is known as **resolution of velocities.**

11.4 To find the resultant of two velocities

Let the moving point *A* possess simultaneously two velocities *u* and *v* which are represented, in both magnitude and direction, by *AB* and *AC* respectively. We complete the parallelogram *ABDC*. By the parallelogram of velocities, the diagonal *AD* represents the resultant velocity *w* in both magnitude and direction. We shall find out the magnitude and direction of w in terms of u and v.

DE is drawn perpendicular to AB (or AB produced).

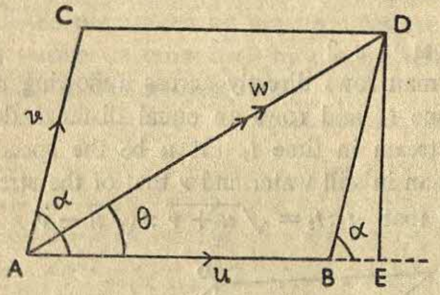


Fig. 11.4

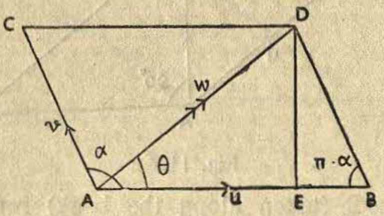


Fig. 11.5

Let $\angle BAC = \alpha$ and $\angle BAD = \theta$.

CASE I: If α be an acute angle (Fig. 11.4), we have

$$DE = BD \sin \alpha = v \sin \alpha$$

and $BE = BD \cos \alpha = v \cos \alpha$.

$\therefore\ AE = AB + BE = u + v \cos \alpha$.

Now, $AD^2 = AE^2 + DE^2$

Or, $w^2 = (u + v \cos \alpha)^2 + (v \sin \alpha)^2$

$$= u^2 + 2uv \cos \alpha + v^2\ (\cos^2 \alpha + \sin^2 \alpha)$$

$$= u^2 + 2uv \cos \alpha + v^2$$

$$[\because\ \cos^2 \alpha + \sin^2 \alpha = 1]$$

$$\therefore\ w = \sqrt{u^2 + 2uv \cos \alpha + v^2} \qquad (1)$$

This gives the magnitude of the resultant velocity w.

Again, $\tan \theta = \dfrac{DE}{AE} = \dfrac{v \sin \alpha}{u + v \cos \alpha}$

$$\therefore\ \theta = \tan^{-1} \left(\frac{v \sin \alpha}{u + v \cos \alpha}\right) \qquad (2)$$

This gives the direction of w.

CASE II: If α be an obtuse angle (Fig. 11.5), we have

$$DE = BD \sin (\pi - \alpha) = v \sin \alpha$$

and $EB = BD \cos (\pi - \alpha) = -\, v \cos \alpha$.

$\therefore\ AE = AB - EB = u$

$$-\,(-\,v \cos \alpha) = u + v \cos \alpha$$

$\therefore\ AD^2 = AE^2 + DE^2$ gives as before,

$$w = \sqrt{u^2 + v^2 + 2uv \cos \alpha}.$$

Also, θ is the same as in (2).

COROLLARIES

1. If $\alpha = 0$, we have $w = u + v$ from (1) and if $\alpha = \pi$, we have $w = u - v$ from (1).

Hence the resultant of two simultaneous velocities along the same line is their algebraic sum.

2. When $\alpha = 90°$, $w = \sqrt{u^2 + v^2}$ from (1).

3. When $u = v$, we have from (1)

$$w^2 = u^2 + 2u^2 \cos \alpha + u^2$$

$$= 2u^2 + 2u^2 \cos \alpha$$

$$= 2u^2 (1 + \cos \alpha)$$

$$= 4u^2 \cos^2 \frac{\alpha}{2}$$

$$\therefore\ w = 2u \cos \frac{\alpha}{2}$$

Also, $\theta = \tan^{-1} \dfrac{u \sin \alpha}{u + u \cos \alpha}$

$$= \tan^{-1} \frac{\sin \alpha}{1 + \cos \alpha}$$

$$= \tan^{-1} \frac{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{2 \cos^2 \frac{\alpha}{2}} = \tan^{-1} \tan \frac{\alpha}{2}$$

$$= \frac{\alpha}{2}$$

Thus the resultant of two equal simultaneous velocities u inclined at an angle α is $2u \cos \frac{\alpha}{2}$ and it acts in a direction bisecting the angle between them.

11.5 Illustrative examples

EXAMPLE 1

A tram-car is moving along a road at the rate of 15 miles per hour; in what direction must a body be projected from it with a velocity of 27 ft./sec., so that its resultant motion may be at right angles to the tram-car ?

Find also the resultant velocity of the body.

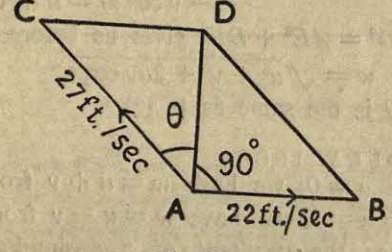


Fig. 11.6

1 mile per hour

$$= \frac{1\times \overset{22}{\cancel{1760}} \times \cancel{3}}{\underset{\underset{15}{\cancel{1200}}}{\cancel{3600}}} \text{ ft./sec.}$$

$$=\frac{22}{15} \text{ ft./sec.}$$

$\therefore$ $15 \text{ m.p.h.} = \left(15 \times \frac{22}{15}\right) \text{ft./sec.} = 22 \text{ ft./sec.}$

Let AB represent the velocity of the tram-car and AC that of the body. We complete the parallelogram $ABDC$ so that AD represents the resultant velocity of the body and is perpendicular to AB.

We have from the right-angled triangle ABD,

$$AD^2 = BD^2 - AB^2 = AC^2 - AB^2 = 27^2 - 22^2$$
$$= 729 - 484$$
$$= 245$$

$\therefore$ $AD = \sqrt{245} = 7\sqrt{5}$

$\therefore$ The resultant velocity of the body is $7\sqrt{5}$ ft./sec.

If $\angle CAD = \theta$, we have from $\triangle CAD$,

$$\cos\theta = \frac{AD}{AC} = \frac{7\sqrt{5}}{27}.$$

Now $\angle BAC = \angle BAD + \angle CAD = 90° + \theta$

$\therefore$ $\sin BAC = \sin(90° + \theta) = \cos\theta = \frac{7\sqrt{5}}{27}$

$\therefore$ $\angle BAC = \sin^{-1}\left(\frac{7\sqrt{5}}{27}\right)$

Hence the body must be projected at an angle $\sin^{-1}\left(\frac{7\sqrt{5}}{27}\right)$ with the direction of motion of the tram-car.

EXAMPLE 2

A man rows directly across a flowing river in time t_1, and rows an equal distance down the stream in time t_2. If u be the speed of the man in still water and v that of the stream, show that $t_1 : t_2 = \sqrt{u+v} : \sqrt{u-v}$.

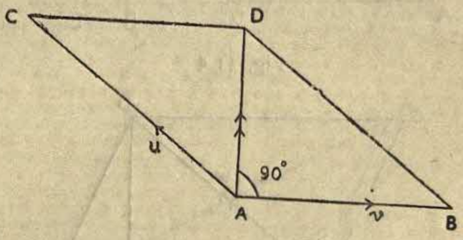


Fig. 11.7

Let AB (taken along the bank) represent the speed v of the stream and AC represent the speed u of the man ; AC is inclined to AB at such an angle that the diagonal AD of the parallelogram $ABDC$ is perpendicular to AB. By parallelogram of velocities, AD represents the resultant velocity of the man. AD being perpendicular to AB, we find that the man rows directly across the river.

$\therefore$ $AD = \sqrt{AC^2 - CD^2} = \sqrt{u^2 - v^2}$ is the resultant velocity of the man.

If d be the breadth of the river, we have

$$t_1 = \frac{d}{\sqrt{u^2 - v^2}}.$$

Again, resultant velocity of the man down the stream is $(u + v)$.

$\therefore$ t_2 = time taken to row a distance d downstream $= \frac{d}{u+v}$.

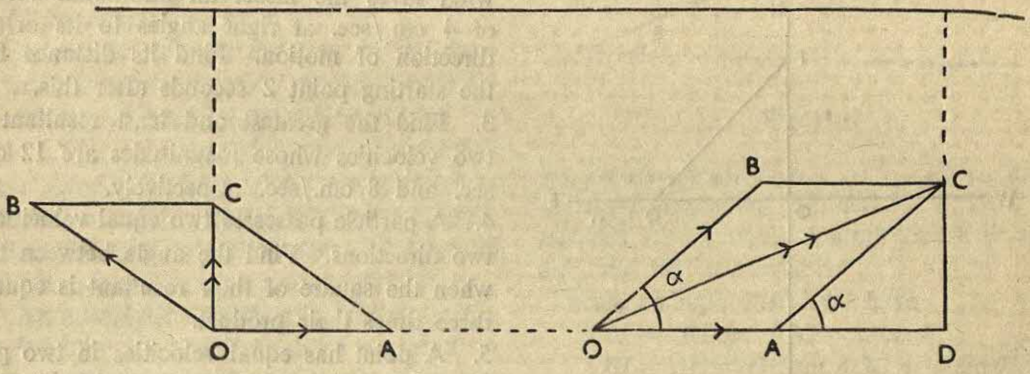
$$\therefore \quad \frac{t_1}{t_2} = \frac{d/\sqrt{u^2 - v^2}}{d/(u+v)} = \frac{u+v}{\sqrt{u^2 - v^2}} = \frac{\sqrt{u+v}}{\sqrt{u-v}}$$

$\therefore$ $t_1 : t_2 = \sqrt{u+v} : \sqrt{u-v}$

Hence the result.

EXAMPLE 3

A man rows across a flowing river with a breadth of 880 yards and a speed of 5 m.p.h., while the stream flows at the rate of 3 m.p.h. Find the time taken by him to cross the river if he wishes to cross it (i) by the shortest path and (ii) in the shortest time.



(a) Fig. 11.8 (b)

$$5 \text{ m.p.h.} = 5 \times \frac{22}{15} \text{ ft./sec.} = \frac{22}{3} \text{ ft./sec.}$$

$$5 \text{ m.p.h.} = 3 \times \frac{22}{15} \text{ ft./sec.} = \frac{22}{5} \text{ ft./sec.}$$

Let OA (along the bank of the river) represent the speed $\frac{22}{5}$ ft./sec. of the stream and OB represent the speed of the man.

(i) When the man crosses the river by the shortest path, the direction of his resultant motion is perpendicular to OA. Hence OB should be inclined to OA at such an angle that the diagonal OC of the parallelogram $OACB$ is perpendicular to OA [Fig. 11.8(a)].

$$\text{Now, } OC = \sqrt{OB^2 - BC^2}$$

$$= \sqrt{\left(\frac{22}{3}\right)^2 - \left(\frac{22}{5}\right)^2}$$

$$= 22\sqrt{\frac{1}{9} - \frac{1}{25}}$$

$$= 22 \times \frac{4}{3\times 5} = \frac{88}{15}$$

∴ The resultant velocity of the man $= \frac{88}{15}$ ft./sec.

Hence the time taken to cross the river

$$= \frac{880\times 3}{\frac{88}{15}} \text{ sec} = \frac{\overset{10}{\cancel{880}}\times 3\times 15}{\cancel{88}} \text{ sec.} = 450 \text{ sec.}$$

$= 7\frac{1}{2}$ minutes.

(ii) Let the man start at any angle α with OA. We complete the parallelogram $OACB$ and draw CD perpendicular to OA produced [Fig. 11.8(b)]. The figure represents the situation in unit time. The man actually crosses the portion CD of the breadth of the river in unit time.

∴ Time taken to cross the river is

$$T = \frac{880}{CD} \text{ secs.} = \frac{880}{AC \sin \alpha}$$

Here T will be least when $\sin\alpha$ is greatest, i.e. when $\alpha = 90°$.

Hence the man himself should start rowing in a direction perpendicular to the current if he wishes to cross the river in shortest time. Then the time taken is

$$T = \frac{880}{AC} \text{ secs. } (\because \sin 90° = 1)$$

$$= \frac{880}{\frac{22}{3}} \text{ secs.} = \frac{\overset{40}{\cancel{880}}\times 3}{\cancel{22}} \text{ secs.} = 120 \text{ secs.}$$

$$= 2 \text{ minutes}$$

EXAMPLE 4

A destroyer steaming north at the rate of

15 m.p.h. observes a sea-plane carrier due east of itself at a distance of 10 miles, the latter steaming due west at the rate of 20 m.p.h. After what time are they at the least distance from one another and what is the least distance ?

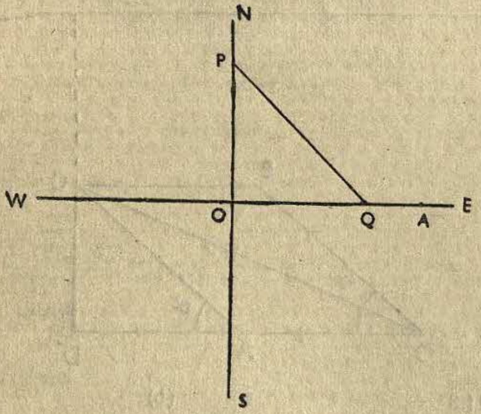


Fig. 11.9

At the initial moment, let O be the position of the destroyer when the sea-plane carrier is at A so that $OA = 10$ miles.

At any time t (hours), let P and Q be the positions of the destroyer and the sea-plane carrier respectively where $PQ = d$ miles.

$\therefore \quad OP = 15t$ miles and $AQ = 20t$ miles.

$\therefore \quad OQ = (10 - 20t)$ miles.

$$\begin{aligned}\text{Now} \quad d^2 = PQ^2 &= OP^2 + OQ^2 \\ &= 225t^2 + (10 - 20t)^2 \\ &= 625t^2 - 400t + 100 \\ &= (625t^2 - 400t + 64) + 36 \\ &= (25t - 8)^2 + 6^2\end{aligned}$$

$\therefore \quad d$ will be least when $25t - 8 = 0$

i.e. when $t = \frac{8}{25}$ hour $= 19\frac{1}{5}$ minutes

and then $d = 6$ miles.

$\therefore$ The destroyer and the carrier are at the least distance of 6 miles after $19\frac{1}{5}$ minutes.

Exercise 11.1

1. A thief, being chased, runs a distance of 3 km. westwards at the rate of 12 km. per hour and then runs northwards a distance of 4 km. at the rate of 16 km. per hour. What are the average speed and the average velocity of the thief for the whole journey ?

2. An insect crawls along a straight line with a velocity of 3 cm./sec. After 3 seconds, wind gives the insect an additional velocity of 4 cm./sec. at right angles to its original direction of motion. Find its distance from the starting point 2 seconds after this.

3. Find the greatest and least resultants of two velocities whose magnitudes are 12 cm./sec. and 8 cm./sec. respectively.

4. A particle possesses two equal velocities in two directions. Find the angle between them when the square of their resultant is equal to three times their product.

5. A point has equal velocities in two given directions. If one of these velocities be halved, the angle which the resultant makes with the other is halved also. Find the angle between the given directions.

6. A particle possesses three simultaneous velocities 7, 5 and 8 cm./sec. acting in different directions and it is noticed that the particle does not move. Find the angle between the latter pair of velocities.

7. Assuming that the bat strikes a cricket ball at an angle of 120°, find the velocity that must be communicated by the bat to the cricket ball travelling horizontally in the line of wickets at 50 ft./sec., so as to make it travel at right angles to its original path.

8. A thief when detected, jumps out of a running train at right angles to its direction with a velocity of 5 ft./sec. If the velocity of the train be 30 m.p.h., find in which direction the thief falls.

9. A swimmer swims directly across a river 450 ft. wide in 3 minutes when there is no current and in 5 minutes when there is current. Find the velocity of the current.

10. A man wishes to row across a river to an exactly opposite point on the other bank ;

if he can row his boat with thrice the velocity of the current, find at what inclination to the current he must keep the boat pointed.

11. A swimmer can swim in still water at the rate of 4 m.p.h. He wishes to cross a river flowing along a straight course at the rate of 2 m.p.h., so as to reach the directly opposite point on the other bank. In what direction should he attempt to swim?

If he wishes to cross the river in the shortest time, in what direction should he swim?

12. A stream runs with a velocity of $1\frac{1}{2}$ m.p.h. Find in what direction a swimmer, whose velocity is $2\frac{1}{2}$ m.p.h., should start in order to cross the stream by the shortest path.

What direction should be taken in order to cross the stream in the shortest time?

13. An aeroplane travelling in still air at the rate of 125 m.p.h., starts from an airport P to reach another airport Q due north of it, 300 miles away. There is a wind blowing due west at the rate of 35 m.p.h., but when half the distance has been covered, the velocity of the wind increases to 75 m.p.h., and the aeroplane adjusts its head accordingly so that it continues its course along PQ as before, and reaches Q. Find the time taken over the flight.

14. A boatman starts rowing across a stream $300\sqrt{3}$ ft. wide, always pointing his boat upstream at an angle of 60° with the bank. Find the time taken by him to cross the stream if he rows with a velocity of 12 ft. per sec. and the current flows at the rate of 4 ft./sec.

How far down the stream will the boat reach the opposite bank below the point at which it was originally directed?

11.6 To resolve a given velocity into two perpendicular components

Let us resolve a given velocity w into two components along the directions OX and OY where $\angle XOY = 90°$.

Let the given velocity w be represented by OC in both magnitude and direction, and let it make an angle θ with OX. Let us complete the parallelogram $OACB$ with OC as diagonal and sides along OX and OY. Then OA and OB represent the components of w along OX and OY respectively.

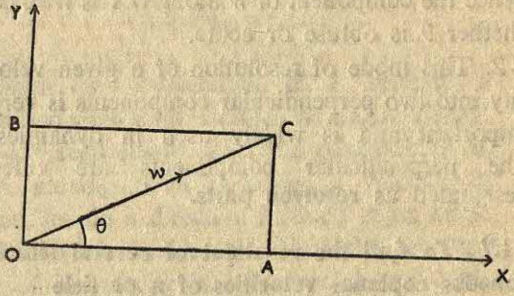


Fig. 11.10

From triangle OAC, we have

$$OA = OC\cos AOC = w\cos\theta,$$
$$OB = AC = OC\sin AOC = w\sin\theta.$$

Hence we conclude that a velocity w is mathematically equivalent to a velocity $w\cos\theta$ in a direction making an angle θ with it, together with a velocity $w\sin\theta$ at right angles to the direction of $w\cos\theta$.

REMARKS

1. Let the angle θ in Fig. 11.10 be obtuse as shown in the Fig. 11.11.

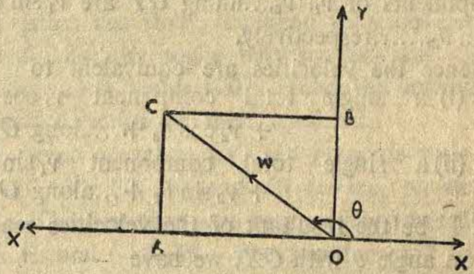


Fig. 11.11

Here the components of w are

$$OA = OC\cos AOC = w\cos(\pi-\theta)$$
$$= -w\cos\theta \text{ along } OX'$$

and $OB = AC = OC\sin AOC$

$$= w\sin(\pi-\theta) = w\sin\theta \text{ along } OY.$$

But $OA = -w\cos\theta$ along OX'

$$= w\cos\theta \text{ along } OX$$

Hence the component of w along OX is $w\cos\theta$ whether θ is obtuse or acute.

2. This mode of resolution of a given velocity into two perpendicular components is very important and is widely used in dynamics. The perpendicular components are often designated as **resolved parts.**

11.7 To find the resultant of several simultaneous coplanar velocities of a particle

Let a point O possess simultaneously the coplanar velocities v_1, v_2,......in directions making angles θ_1, θ_2......respectively with a fixed line OX in the plane of the velocities. Let us take a line OY at right angles to OX at O.

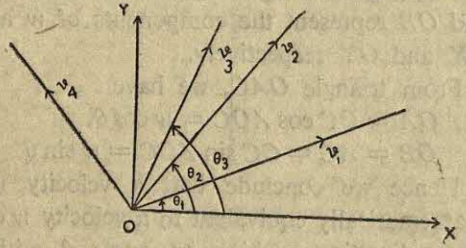


Fig. 11.12

The components of v_1, v_2,......along OX are $v_1\cos\theta_1$, $v_2\cos\theta_2$,......respectively, while the components of v_1, v_2,...along OY are $v_1\sin\theta_1$, $v_2\sin\theta_2$,......respectively.

Hence the velocities are equivalent to

(i) a single total component $v_1\cos\theta_1 + v_2\cos\theta_2 + \ldots$ along OX

and (ii) a single total component $v_1\sin\theta_1 + v_2\sin\theta_2 + \ldots$ along OY

If V be the resultant of the velocities making an angle θ with OX, we have

$$V\cos\theta = v_1\cos\theta_1 + v_2\cos\theta_2 + \ldots = \Sigma\, v_1\cos\theta_1 \qquad (1)$$

$$\text{and } V\sin\theta = v_1\sin\theta_1 + v_2\sin\theta_2 + \ldots = \Sigma\, v_1\sin\theta_1 \qquad (2)$$

Squaring (1) and (2) and adding,

$$V^2 = (v_1\cos\theta_1 + v_2\cos\theta_2 + \ldots)^2 + (v_1\sin\theta_1 + v_2\sin\theta_2 + \ldots)^2$$

$$= \Sigma\, v_1^2 + 2\ \Sigma\, v_1 v_2\cos(\theta_2 \sim \theta_1)$$

so that $V = \sqrt{\Sigma\, v_1^2 + 2\ \Sigma\, v_1 v_2 \cos(\theta_2 \sim \theta_1)}$ (3)

Here the positive sign of the square root is taken to denote the magnitude of V. The direction of V is determined by the signs of $\cos\theta$, $\sin\theta$ in (1) and (2).

Dividing (2) by (1), we get

$$\theta = \tan^{-1}\frac{\Sigma\, v_1\sin\theta_1}{\Sigma\, v_1\cos\theta_1} \qquad (4)$$

which gives the direction of the resultant velocity V.

EXAMPLE

A particle possesses the simultaneous velocities 1, 2, $3\sqrt{3}$ and 4 ft./sec. towards the east, 60° north of east, 30° north of west and 60° south of east respectively. Find the displacement of the particle 10 secs. after start.

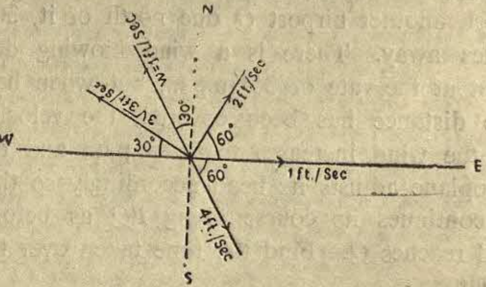


Fig. 11.13

Let u, v be the components of the resultant velocity w towards east and north respectively, and let w act at an angle θ north of east.

Then $u = w\cos\theta$, $v = w\sin\theta$ (1)

and also, $u = 1 + 2\cos 60^\circ - 3\sqrt{3}\cos 30^\circ + 4\cos 60^\circ$

$$= 1 + 1 - \tfrac{9}{2} + 2 = -\tfrac{1}{2} \qquad (2)$$

and $v = 0 + 2\sin 60^\circ + 3\sqrt{3}\sin 30^\circ - 4\sin 60^\circ$

$$= \sqrt{3} + \frac{3\sqrt{3}}{2} - 2\sqrt{3} = \frac{\sqrt{3}}{2} \qquad (3)$$

[The first velocity has no component along the north. The second is equivalent to components $2\cos 60^\circ$ and $2\sin 60^\circ$ towards east and north respectively.

The third velocity has components

$3\sqrt{3}\cos 30^\circ$ and $3\sqrt{3}\sin 30^\circ$ along west and north.

i.e. $-3\sqrt{3}\cos 30^\circ$ and $3\sqrt{3}\sin 30^\circ$ along east and north.

The fourth velocity has components

$4\cos 60^\circ$ and $4\sin 60^\circ$ along east and south.

i.e. $4\cos 60^\circ$ and $-4\sin 60^\circ$ along east and north]

By (1), (2) and (3), we have

$$w^2 = u^2 + v^2 = \tfrac{1}{4} + \tfrac{3}{4} = 1 \text{ ; or, } |w| = 1$$

and $\tan\theta = \dfrac{v}{u} = -\sqrt{3} = \tan 120^\circ$;

or, $\theta = 120^\circ$

Hence the resultant velocity is 1 ft./sec. at an angle 120° with the east, i.e. towards 30° west of north.

$\therefore$ Displacement of the particle after 10 secs. = 10 ft. towards 30° west of north.

11.8 Triangle of velocities

If a moving point possesses simultaneously two velocities represented in magnitude, direction and sense by the two sides of a triangle taken in order, their resultant is represented by the third side taken in opposite order.

Let AB and AD represent the simultaneous velocities of the point A. The parallelogram $ABCD$ is completed. By parallelogram of velocities, AC represents the resultant of the velocities AB and AD, i.e. of AB and BC. But the order of the resultant velocity AC is opposite to the order in which AB and BC are considered. Hence the proposition.

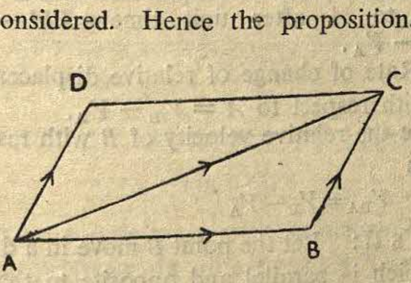


Fig. 11.14

Using vector notations, the proposition may be written as $\overline{AB} + \overline{BC} = AC$.

Exercise 11.2

1. Find the components of a velocity u along two directions inclined at angles of 30° and 45° respectively with the direction of u.
2. A moving point possesses a velocity of 10 cm./sec. in a direction inclined at an angle of 30° to the horizontal. Find the resolved parts of its velocity in a horizontal and vertical direction respectively.
3. Find a horizontal velocity and a velocity inclined at an angle of 60° with the vertical whose resultant shall be a given velocity 7 ft./sec. in the vertical direction.
4. A particle has three simultaneous velocities 5, 3 and 4 ft./sec. The angle between the first and the second is 90° while the direction of the third bisects the angle between the former two. Find the magnitude and direction of a fourth velocity which must be imparted to it so that it may remain at rest.
5. A point possesses three simultaneous velocities in one plane, whose magnitudes are 2, $2\sqrt{2}$ and 1 ft./sec. respectively; the first is horizontal, the second acts at 45° to the horizon, and the third is vertical; find their resultant.
6. A point has three coplanar velocities $5u$, $10u$ and $13u$, the angle between any two of their directions being 120°. Find the magnitude and direction of their resultant.
7. A point possesses five simultaneous velocities 10, 20, 30, 40 and 50 ft./sec. respectively. The first three are respectively towards E., N. E. and S. S. W. The fourth is 15° W. of N., and the fifth 30° E. of S. Find the displacement of the particle after 5 secs. after start.
8. A point possesses simultaneously velocities represented by u, $2u$, $3\sqrt{3}\,u$ and $4u$; the angles between the first and second, the second

and third, the third and fourth are respectively 60°, 90° and 150°; show that the resultant velocity is u in a direction inclined at an angle of 120° to that of the first velocity.

9. A particle possesses simultaneously three velocities u, v, w in directions inclined at angles α, β, γ with one another; show that the resultant velocity is $\{u^2 + v^2 + w^2 + 2uv\cos\alpha + 2vw\cos\beta + 2wu\cos\gamma\}^{\frac{1}{2}}$.

10. A point has velocities represented by OA and OB. Show that its resultant velocity may be represented by $2OG$ where G is the mid-point of AB.

11. The sides AB and AC of a triangle ABC are bisected in D and E. Show that the resultant of velocities represented by BE and DC is represented in magnitude and direction by $\frac{3}{2}BC$.

11.9 Relative motion

Whenever we speak of velocity of a moving point A, we mean the rate of change of its position with reference to some point O which is regarded as fixed. The motion of A, in this case, is said to be absolute. In fact no motion observed by us is absolute because the earth herself is moving round the sun. However, we treat the motion as absolute when it is taken with reference to some frame of reference fixed on the earth. Now if the reference point O also moves, it would be necessary to know the motion of O with reference to some other point (fixed). In this case, the motion of A observed from O is not absolute.

A man sitting on a running train sees the trees, houses, etc. outside as running past the train in the opposite direction while the train itself seems to be stationary. The objects outside the moving train are actually fixed and their motion observed by the man is their relative motion with respect to the train.

11.10 Relative velocity

The rate of change of position of a moving point A with respect to some other moving point B is called the **relative velocity** of A with respect to B.

We shall learn in the following paragraph how to determine the relative velocity of a moving point with respect to some other moving point.

11.11 To determine the relative velocity of a moving point B with respect to some other moving point A

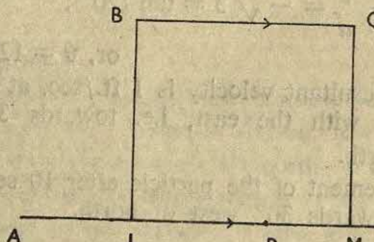


Fig. 11.15

CASE I: Let the moving points A and B move with velocities V_A and V_B respectively in parallel directions AM and BQ. Initially, they start from A and B; and let P, Q be their positions at the end of unit time.

$$\therefore \quad V_A = AP \text{ and } V_B = BQ \tag{1}$$

Now, the difference in distance between the moving points after unit time

$$\begin{aligned} &= PM \\ &= AM - AP \\ &= AL + LM - AP \\ &= AL + BQ - AP \\ &= AL + V_B - V_A \text{ by (1)} \end{aligned}$$

$\therefore$ Change in relative displacement of B with respect to A after unit time $= PM - AL = V_B - V_A$.

$\therefore$ Rate of change of relative displacement of B with respect to $A = V_B - V_A$.

Hence the relative velocity of B with respect to A is

$$V_{BA} = V_B - V_A \tag{2}$$

CASE II: Let the point B move in a direction which is parallel and opposite to that of A.

Replacing V_B by $-V_B$, we get

$$V_{BA} = -(V_A + V_B) \qquad (3)$$

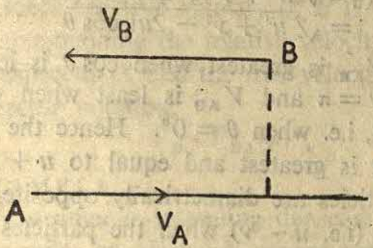


Fig. 11.16

CASE III: Let both the points A and B move with equal velocities in the same direction. In this case, we find $V_{BA} = 0$ putting $V_A = V_B$ in (2). Hence B will appear to A to be at rest and their relative velocity with respect to each other is nil.

CASE IV: Let A and B both move with the same velocity in opposite directions.

Putting $V_B = -V_A$ in (3),
we have $V_{BA} = -2V_A = -2V_B$ (4)
Hence B will appear to A to be moving with twice its speed in the opposite direction, and vice versa.

CASE V: Let B move in a direction which is inclined to the direction of motion of A at some angle θ.

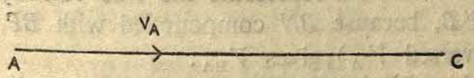


Fig. 11.17

Let A and B move with velocities V_A and V_B respectively in directions AC and BD where BD is inclined to the direction of AC at an angle θ.

Now V_B can be resolved into two components:

(i) $V_B \cos\theta$ parallel to AC
and (ii) $V_B \sin\theta$ perpendicular to AC.

The components of V_A are

(iii) V_A along AC and (iv) 0 perpendicular to AC.

Taking (i) and (iii), we may think of A and B moving with velocities V_A and $V_B \cos\theta$ respectively in the same direction AC.

$\therefore$ Component of relative velocity B with respect to A along $AC = V_B \cos\theta - V_A$ by Case I.

Similarly taking (ii) and (iv), component of relative velocity of B with respect to $A = V_B \cdot \sin\theta - 0 = V_B \sin\theta$ by Case I.

Hence, if the relative velocity V_{BA} makes an angle α with the direction AC, we have

$$V_{BA} \cos\alpha = V_B \cos\theta - V_A \qquad (5)$$

and $V_{BA} \sin\alpha = V_B \sin\theta$ (6)

Squaring and adding (5) and (6), we get

$$V_{BA} = \sqrt{V_A^2 - 2V_A V_B \cos\theta + V_B^2} \qquad (7)$$

The direction of V_{BA} is given by

$$\alpha = \tan^{-1} \frac{V_B \sin\theta}{V_B \cos\theta - V_A} \qquad (8)$$

REMARK

We find that $V_{BA} = V_B - V_A$ is the resultant of V_B and the reversed V_A. But A is at rest when V_A is reversed. Hence the relative velocity of B with respect to A is obtained by bringing A to rest.

COROLLARIES

Let three points A, B, C move simultaneously in the same direction with velocities V_A, V_B and V_C respectively. Then the relative velocity of C with respect to A is

$$\begin{aligned} V_{CA} &= V_C - V_A \\ &= (V_C - V_B) + (V_B - V_A) \\ &= V_{CB} + V_{BA} \end{aligned}$$

Hence the relative velocity of C with respect to A is obtained by compounding the relative velocities of C with respect to B and B with respect to A.

EXAMPLE

Two particles move with speeds u and v respectively in opposite sense along the circumference of a circle. In what positions will their relative veolcity be greatest and least, and what will be its values then ?

What would happen if they move in the same sense ?

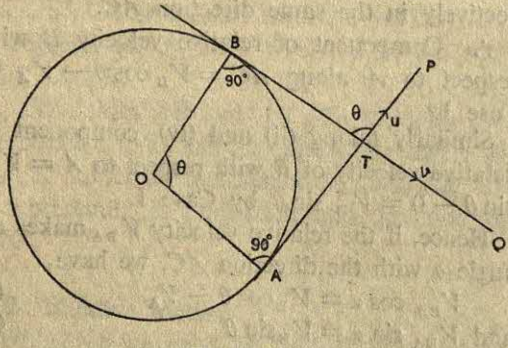


Fig. 11.18

Let the two particles move on the circumference of the circle of centre O and let A, B be their positions at any time t where $\angle AOB = \theta$. The velocities u and v at the instant t act along the tangents ATP and BTQ respectively.

Since $\angle OAT = \angle OBT = 90°$, we have $\angle PTB = \angle AOB = \theta$.

The relative velocity of A with respect to B, i.e. V_{AB} is the resultant of u along TP and v along TB (i.e. the reversed v).

$$\therefore \quad V_{AB} = \sqrt{u^2 + v^2 + 2uv\cos\theta} \qquad (1)$$

Now V_{AB} is greatest when $\cos\theta$ is greatest, i.e. $\cos\theta = 1$, or $\theta = 0°$; i.e. when the particles meet each other.

$\therefore$ Greatest value of $V_{AB} = u + v$ [putting $\cos\theta = 1$ in (1)].

Again, V_{AB} is least when $\cos\theta$ is least, i.e. $\cos\theta = -1$, or, $\theta = \pi$, i.e. when the particles are diametrically opposite.

$\therefore$ Least value of $V_{AB} =$

$$\sqrt{u^2 + v^2 - 2uv} = u \sim v.$$

If the particles move in the same sense, v acts along QTB and therefore, reversed v acts along TQ. Now $\angle PTQ = 180° - \theta$.

$$\therefore \quad V_{AB} = \sqrt{u^2 + v^2 + 2uv\cos(180° - \theta)}$$
$$= \sqrt{u^2 + v^2 - 2uv\cos\theta}$$

Here V_{AB} is greatest when $\cos\theta$ is least, i.e. when $\theta = \pi$ and V_{AB} is least when $\cos\theta$ is greatest, i.e. when $\theta = 0°$. Hence the relative velocity is greatest and equal to $u + v$ when the particles are diametrically opposite, and it is least (i.e. $u \sim v$) when the particles meet.

11.12 To determine the true velocity when the relative velocity is known

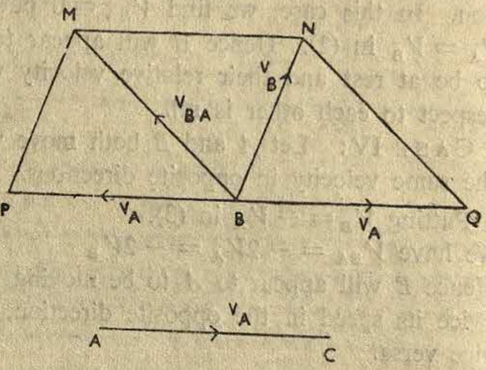


Fig. 11.19

We are given the relative velocity V_{BA} of a point B with respect to A ; we have to find out the true velocity V_B of B. Let the true velocity V_A of A be represented by AC and the relative velocity V_{BA} by BM. BP is drawn equal and parallel to AC in the opposite sense. The parallelogram $BPMN$ is completed with BM as diagonal and BP as a side. The other side BN then represents the true velocity V_B of B, because BN compounded with BP (the reversed V_A) gives V_{BA}.

Let PB be now produced to Q such that $BQ = AC$. Then BQ represents V_A in magnitude and direction. It is obvious that $BMNQ$ is a parallelogram with BN as its diagonal. By parallelogram of velocities, the velocity represented by BN is the resultant of the velocities represented by *BQ* and *BM*, i.e. V_B is the resultant of V_A and V_{BA}.

Hence the true velocity of *B* is obtained by compounding the relative velocity of *B* with respect to *A* and the true velocity of *A*. In vector notation,

$$\bar{V}_B = \bar{V}_{BA} + \bar{V}_A$$

EXAMPLE

A steamer is travelling due east at the rate of *u* miles an hour. A second steamer is travelling at $2u$ miles an hour in a direction θ north of east, and appears to be travelling north-east to a passenger on the first steamer. Prove that $\theta = \frac{1}{2}\sin^{-1}\frac{3}{4}$.

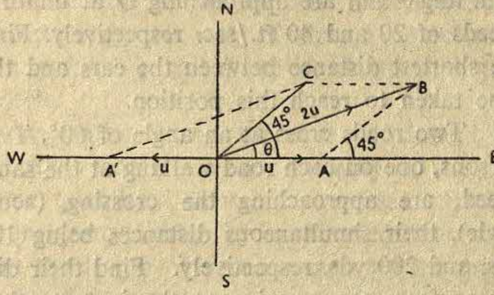


Fig. 11.20

Let the velocity *u* of the first steamer be represented by *OA* and the velocity $2u$ of the second steamer be represented by *OB* where $\angle AOB = \theta$. Let *OC* represent the apparent (relative) velocity of the second steamer where $\angle AOC = 45°$. By Section 11.12, *OB* is the diagonal of the parallelogram *OABC*.

Now $\angle OAB = 180° - \angle BAE = 180° - 45° = 135°$

and $\angle OBA = \angle BOC = 45° - \theta$.

By trigonometry, we have from $\triangle OAB$

$$\frac{\sin OBA}{OA} = \frac{\sin OAB}{OB}$$

or, $$\frac{\sin(45° - \theta)}{u} = \frac{\sin(\pi - 45°)}{2u}$$

or, $$\frac{\sin(45° - \theta)}{\sin 45°} = \frac{u}{2u} = \frac{1}{2}$$

or, $$\frac{\sin 45° \cos\theta - \cos 45° \sin\theta}{\sin 45°} = \frac{1}{2}$$

or, $\cos\theta - \sin\theta = \frac{1}{2}$

Squaring this relation, we get

$1 - \sin 2\theta = \frac{1}{4}$

or, $\sin 2\theta = 1 - \frac{1}{4} = \frac{3}{4}$

or, $2\theta = \sin^{-1}\frac{3}{4}$

$\therefore\ \theta = \frac{1}{2}\sin^{-1}\frac{3}{4}$.

Exercise 11.3

1. A ship is sailing due east at 8 m.p.h. and the wind is blowing from the north-west at $8\sqrt{2}$ m.p.h. Show that, to an observer on the ship, wind appears to come from the north at 8 m.p.h.

2. A person travels due east at the rate of 4 m.p.h. and the wind blows from the north-west at the rate of $4\sqrt{2}$ m.p.h. Find in magnitude and direction the velocity of the wind relative to the person.

What happens if the person doubles his speed without altering his direction of motion ?

3. A railway train, moving at the rate of 30 m.p.h. is struck by a stone, moving horizontally and at right angles to the train with a velocity of 33 ft./sec. Find the magnitude and direction of the velocity with which the stone appears to meet the train.

4. One boat is sailing north at the rate of 12 m.p.h. and another boat is sailing north-west at the rate of $12\sqrt{2}$ m.p.h. Find in magnitude and direction the velocity of the second boat relative to the first.

5. A train is travelling north at 60 m.p.h. and the wind is blowing from the south-west at 20 m.p.h. Find the direction of the trail of the smoke of the engine.

[Assume that the smoke loses the velocity of the train as soon as it leaves the funnel and moves with the velocity of the wind.]

6. A school boy holding an umbrella runs with a velocity equal to that of the rain falling vertically, in consequence of which the

rain strikes him in the face. At what angle should he hold the umbrella in order to protect him best ?

7. A north wind is blowing at the rate of 8 m.p.h. To a cyclist it appears to be an east wind of 8 m.p.h. Find the velocity of the cyclist in direction and magnitude.

8. To a cyclist travelling at 10 m.p.h. due east, the wind appears to come from the north-east ; but when he travels north-east at the same speed it appears to come from the north. Find the true direction and velocity of the wind.

9. To a passenger of a train moving at 45 m.p.h. through rain falling vertically, the direction of the drops appears to make an angle $\tan^{-1}(\frac{3}{2})$ with the vertical ; find the velocity of the rain.

10. To a motor-car passenger travelling due south at 30 m.p.h. the wind appears to blow from the west. When the motor-car speed is reduced to 15 m.p.h., the wind appears to come from the north-west. Find the true velocity and direction of the wind.

11. To a man walking at the rate of 2 m.p.h. the rain appears to fall vertically ; when he increases his speed to 4 m.p.h. it appears to meet him at an angle of 45° ; find the real direction and speed of the rain.

12. To a man walking at the rate of 3 m.p.h. the rain appears to fall vertically ; find the real direction of the rain, given that its apparent velocity is $3\sqrt{3}$ m.p.h.

13. A person going eastward with a speed of 4 m.p.h. finds that the wind appears to blow directly from the north. He doubles his speed and the wind seems to come from the north-east ; find the direction of the wind and its velocity.

14. A ship steams due west at the rate of 15 m.p.h. relative to the current which is flowing at the rate of 6 m.p.h. due south. What is the velocity relative to the ship, of a train going due north at the rate of 30 m.p.h. ?

15. At a given instant two cars are at distances 300 and 400 yds. from the point of intersection O of two straight roads crossing at right angle and are approaching O at uniform speeds of 20 and 40 ft./sec. respectively. Find the shortest distance between the cars and the time taken to reach this position.

16. Two roads cross at an angle of 60°. Two persons, one on each road walking at the same speed, are approaching the crossing (acute angle), their simultaneous distances being 100 yds. and 200 yds. respectively. Find their distances from the crossing at the instant when they are nearest to one another.

17. One ship, sailing east with a speed of 15 m.p.h. passes a certain point at noon ; and a second ship sailing north at the same speed, passes the same point at 1.30 P.M. At what time are they closest together and what is the distance then between them ?

12 Acceleration

WE HAVE seen in the previous chapter that the displacement of a moving particle changes with time and the time-rate of change of displacement has been defined as the **velocity** of the particle.

The velocity of a particle may also change in the course of its motion. The quantity characterising the rate of change of velocity of a particle is called the **acceleration of the particle.**

Acceleration being the rate of change of velocity which is a vector quantity, is itself a vector quantity. Hence we represent acceleration geometrically by a directed line segment the length of which stands for the magnitude of the acceleration.

If the velocity of a moving particle increases, the motion is said to be **accelerated;** if the velocity decreases, the motion is said to be **retarded** or **decelerated.** In other words, acceleration is positive in case of an accelerated motion and it is negative in case of a retarded motion. Negative acceleration of a particle is frequently termed as **retardation** or **deceleration.**

The acceleration of a particle is said to be **uniform** if its velocity undergoes equal changes in the same direction in equal intervals of time, however small these time intervals may be. It is measured by the change in velocity in unit time.

When the acceleration of a particle is not uniform, i.e. when it changes in magnitude or direction or both, we say that the acceleration of the particle is variable; and we measure it at any time by the final change of velocity in a unit of time, if during that time the particle continues to move with the same acceleration as at the instant under consideration.

The units of acceleration are **ft./sec^2** in the F.P.S system and **cm./sec.^{2}** in the C.G.S. system, its dimension being $\mathbf{LT^{-2}}$.

If a particle possesses several simultaneous accelerations in different directions and if their combined effect be the same as if the particle moves with a single acceleration in a definite direction, this latter acceleration is called the **resultant** of the given simultaneous accelerations which, in their turn, are called the **component** accelerations of the single resultant. As in the previous chapter, the method of composition of two given accelerations is provided by what is known as the **parallelogram of accelerations.**

> If a particle possesses simultaneously two uniform accelerations which are represented in magnitude and direction by the two adjacent sides of a parallelogram drawn from a point, then these accelerations can be compounded into a single uniform acceleration which is represented in magnitude and direction by the diagonal of the parallelogram passing through that point. This is known as the parallelogram of acceleration.

12.1 Velocity and acceleration of a particle in rectilinear motion

We shall now discuss the motion of a particle along a straight line, otherwise called the **rectilinear motion** of the particle.

Let us consider a particle P moving in a straight line. A point O on that line is taken as the origin of a coordinate system and the axis OX (the x-axis) is taken along the line. The position of the particle on the axis will be specified by its x coordinate, equal to the distance OP taken with the appropriate sign. The displacement of the particle P carries it through positions P_1, P_2......, i.e. the value of x changes continuously. The position of the particle on its path at any given instant is specified by the relationship

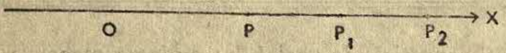


Fig. 12.1

$$x = f(t) \qquad (1)$$

Equation (1) gives the general equation of rectilinear motion.

It should be noted that the quantity x in equation (1) specifies the *position* of a moving particle, not the distance travelled by it. Thus, if a particle moving from O reaches position P_1 (see Fig. 12.1) and then reverses its motion and returns to P, its coordinate by that time will be $x = OP$, but the distance travelled by it will be $d = OP_1 + P_1P$ so that $x \neq d$.

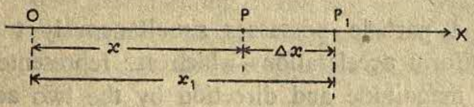


Fig 12.2

Let P, P_1 be the positions of the particle moving on OX at times t, t_1 respectively, its displacement in the time interval $\triangle t = t_1 - t$ being $\triangle x = x_1 - x$ (see Fig. 12.2). The average velocity of the particle is a vector quantity $\vec{v}_{av}$ of which the numerical value is

$$v_{av} = \frac{x_1 - x}{t_1 - t} = \frac{\triangle x}{\triangle t} \qquad (2)$$

Here $\vec{v}_{av}$ has the same direction as the displacement of the particle, i.e. from P to P_1.

The less the interval of time $\triangle t = t_1 - t$ for which the average velocity is calculated, the more accurately will the quantity v_{av} characterise the motion of the particle. In order to obtain a characteristic of motion which would be independent of the time interval $\triangle t$, the concept of **instantaneous velocity** of a particle is introduced.

The instantaneous velocity of a particle at a time t is the quantity $\vec{v}$ towards which the average velocity $\vec{v}_{av}$ tends when the time interval $\triangle t \rightarrow 0$. The numerical value of v is, therefore, given by

$$v = \lim_{\triangle t \rightarrow 0} \frac{\triangle x}{\triangle t} = \frac{dx}{dt} \qquad (3)$$

Hence the numerical value of the instantaneous velocity of a particle moving on a straight line is equal to the first derivative of its coordinate x with respect to time t.

The sign of v is the same as the sign of $\triangle x$. If $v > 0$, the motion of the particle is in the direction of x increasing (the vector $\vec{v}$ is in the direction of the positive x-axis). If $v < 0$, motion of the particle is in the direction of x decreasing (the vector $\vec{v}$ is in the direction of the negative x-axis).

The velocity of a particle may undergo changes in the course of its motion. The time rate of change of velocity of a particle is called **the acceleration of the particle**.

Let us first introduce the concept of **average acceleration.** Let v and v_1 be the velocities of a particle moving on OX at times t and t_1 respectively. The average acceleration of the particle in the time interval $\triangle t = t_1 - t$ is a vector quantity $\vec{f}_{av}$ of which the numerical value is given by

$$f_{av} = \frac{v_1 - v}{t_1 - t} = \frac{\triangle v}{\triangle t} \tag{4}$$

where $\triangle v$ is the increment in the value of v in the time interval $\triangle t$.

$\vec{f}_{av}$ is in the positive direction of the x-axis when $f_{av} > 0$ and in the negative direction when $f_{av} < 0$.

The **instantaneous acceleration** $\vec{f}$ of a particle for any given time t is defined as the limit towards which the average acceleration $\vec{f}_{av}$ tends when the time interval $\triangle t \longrightarrow 0$. The numerical value of $\vec{f}$ is given by

$$f = \lim_{\Delta t \to 0} \frac{\triangle v}{\triangle t} = \frac{dv}{dt}.$$

Using (3), we have

$$f = \frac{dv}{dt} = \frac{d^2 x}{dt^2}. \tag{5}$$

Hence the numerical value of the instantaneous acceleration of a particle is, in rectilinear motion, equal to the first derivative of the velocity, or the second derivative of the coordinate x of the particle with respect to time.

Again, $\frac{dv^2}{dt} = 2\,v\,\frac{dv}{dt} = 2\,vf.$ (6)

We find from (6) that if v, f have the same sign (both positive or both negative), then $\frac{dv^2}{dt} > 0$ and v^2 is increasing. Hence the magnitude of $\vec{v}$ $(= +\sqrt{v^2})$ is increasing when v, f have the same sign. If v, f have opposite signs, the magnitude of $\vec{v}$ is decreasing. Thus, the rectilinear motion of a particle is *accelerated* when the velocity and acceleration vectors are in the same direction ; it is *retarded* when the velocity and acceleration vectors are of opposite sense.

We may rewrite (5) in the form

$$f = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v\frac{dv}{dx} \tag{7a}$$

$$\text{or,} \quad f = \tfrac{1}{2}\,\frac{dv^2}{dx}. \tag{7b}$$

12.2 Rectilinear motion of a particle with uniform acceleration

The rectilinear motion of a particle is described fully when we can determine the distance, the velocity and the acceleration of the particle at any instant. Hence we must have some relations among these quantities. The required relations are as follows:

> If a particle moves along a straight line with a uniform acceleration f, and if u and v be the initial and final velocities during an interval of time t and x be the distance described in time t, then
>
> (i) $v = u + ft$,
> (ii) $x = ut + \frac{1}{2}ft^2$,
> (iii) $v^2 = u^2 + 2fx$.

The proofs of these formulae are given below:

PROOF OF (*i*)

Let a particle move along a straight line with uniform acceleration f and let it start from the position $x = 0$ at $t = 0$ with a velocity u. The velocity v of the particle at any time is given by

$$\frac{dv}{dt} = f, \text{ or } dv = f dt$$

Integrating with respect to t, we get

$$v = ft + A \tag{1}$$

where A is an arbitrary constant.

The initial condition is

$$v = u \text{ when } t = 0. \tag{2}$$

Using (2), we have from (1), $A = u$.

Thus (1) becomes $v = u + ft$ (i)

PROOF OF (*ii*)

The velocity v of the particle at any time t is

$$v = \frac{dx}{dt} = u + ft \quad \text{by (i)},$$

or, $dx = udt + ft\,dt$.

Integrating, we get

$$x = ut + \tfrac{1}{2}ft^2 + B \tag{3}$$

where B is an arbitrary constant.

The initial condition is

$$x = 0 \text{ at } t = 0. \tag{4}$$

Using (4), we have from (3), $B = 0$.

$$\therefore \quad x = ut + \tfrac{1}{2}ft^2. \tag{ii}$$

PROOF OF (*iii*)

We have $v\dfrac{dv}{dx} = f$

or, $vdv = fdx$.

This gives on integration

$$\frac{v^2}{2} = fx + D \tag{5}$$

where D is an arbitrary constant.

The initial condition is

$$v = u \text{ when } x = 0 \text{ (at time } t = 0). \tag{6}$$

By (6), we get from (5), $D = \dfrac{u^2}{2}$.

$\therefore$ (5) becomes $\quad \dfrac{v^2}{2} = \dfrac{u^2}{2} + fx$

$$\text{so that} \quad v^2 = u^2 + 2fx. \tag{iii}$$

COROLLARY

1. Using (ii) above, we can determine the distance described by the particle in any particular second. Thus, if x_t denotes the distance described by the particle in the tth second, we have

x_t = distance described in t secs.

$\quad$ − distance described in $(t-1)$ secs.

$$= ut + \tfrac{1}{2}ft^2 - \left\{u(t-1) + \tfrac{1}{2}f(t-1)^2\right\}$$

$$\text{or,} \quad x_t = u + \tfrac{1}{2}f(2t-1) \tag{iv}$$

2. It can also be shown that *if acceleration be constant, average velocity in any interval of time is equal to the average of velocities at the beginning and end of the interval.*

The average velocity (strictly the time average $\dfrac{\int_{t_1}^{t_2} v\,dt}{\int_{t_1}^{t_2} dt}$ in the time interval from t_1 to t_2) is

$$\frac{\text{distance covered}}{\text{time taken}}.$$

We shall prove the result for the time interval $(0, t_1)$. Let $v = u$ when $t = 0$ and $v = v_1$ when $t = t_1$. If x_1 be the distance described by the particle in time t_1, we have

$$x_1 = ut_1 + \tfrac{1}{2}ft_1^2 \quad \text{by (ii) above,}$$

$$\text{and } v_1 = u + ft_1 \quad \text{by (i) above.}$$

$$\begin{aligned} \therefore \quad \text{Average velocity} &= \frac{x_1}{t_1} = u + \tfrac{1}{2}ft_1 \\ &= u + \tfrac{1}{2}(v_1 - u) \\ &= \frac{u + v_1}{2} \end{aligned}$$

$$= \frac{\text{initial velocity} + \text{final velocity}}{2}.$$

REMARK

If the particle starts from rest (i.e. $u = 0$), the above relations (i)-(iii) become

$$v = ft,$$
$$x = \tfrac{1}{2}ft^2,$$
$$v^2 = 2fx.$$

12.3 Illustrative examples

EXAMPLE 1

A particle starts with an initial velocity u and passes successively over the two-halves of a given distance with accelerations f and f' respectively. Show that the final velocity is the same as if the whole distance is covered with uniform acceleration $\frac{1}{2}(f + f')$.

Let $2x$ be the total distance, v_1 the velocity after a distance x is covered with acceleration f, and v_2 the velocity at the end of the journey.

$$\therefore \quad v_1^2 = u^2 + 2fx. \tag{1}$$

The initial velocity for the second-half of the journey is equal to the final velocity of the first-half, i.e. v_1.

$$\therefore \quad v_2^2 = v_1^2 + 2f'x$$

$$= u^2 + 2fx + 2f'x \text{ by (1)}$$
$$= u^2 + 2.\tfrac{1}{2}(f + f').\,2x. \qquad (2)$$

If the total distance were described with a constant acceleration f_c, we would have

$$v_2^{\,2} = u^2 + 2f_c.2x. \qquad (3)$$

Comparing (2) with (3), we find

$$f_c = \tfrac{1}{2}(f + f').$$

Hence the final velocity is the same as if the whole distance is covered with uniform acceleration f_c.

EXAMPLE 2

A train stopping at two stations 2 miles apart takes 4 minutes on the journey from one of the stations to the other. Assuming that its motion is first that of uniform acceleration x and then that of uniform retardation y, prove that $\frac{1}{x} + \frac{1}{y} = 4$, a mile and a minute being the unit of distance and time respectively.

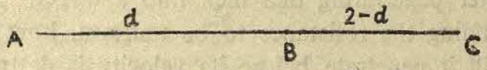


Fig. 12.3

Let the stations be A and C. We suppose that the train moves with uniform acceleration x from A to B, and then it moves with uniform retardation y from B to C where $AB = d$ miles so that $BC = (2 - d)$ miles.

If v be the velocity of the train at B, we have

$$v = xt \qquad (1)$$
$$v^2 = 2xd \qquad (2)$$
$$o = v - y(4 - t)$$
$$\text{or, } v = y(4 - t) \qquad (3)$$
$$\text{and } v^2 = 2y(2 - d), \qquad (4)$$

t being time (in minutes) taken by the train to move from A to B. Here relations (1) and (2) correspond to the motion from A to B, and (3), (4) stand for the motion from B to C.

From (1) & (3), we have

$$v\left(\frac{1}{x} + \frac{1}{y}\right) = t + 4 - t = 4 \qquad (5)$$

Again, (2) and (4) give

$$\frac{v^2}{2}\left(\frac{1}{x} + \frac{1}{y}\right) = d + 2 - d = 2 \qquad (6)$$

Dividing (6) by (5), we get

$$\frac{v}{2} = \frac{1}{2} \text{ so that } v = 1.$$

Hence (5) becomes $\frac{1}{x} + \frac{1}{y} = 4$

Thus the result is proved.

EXAMPLE 3

A man sees a bus 100 yds. away starting from rest with constant acceleration. He then runs after it with constant speed and just catches it in one minute. Determine the speed of the man and the acceleration of the bus.

If the man's speed is 8 ft./sec., find in yards the nearest he can get to the bus.

Let the man run at v ft./sec. and the bus accelerates at f ft./sec^2. Then in t secs. the man has run a distance vt and the bus has moved a distance $\frac{1}{2}ft^2$. The bus is then ahead of the man by a distance d ft. where

$$d = \tfrac{1}{2}ft^2 + 300 - vt$$
$$= \tfrac{1}{2}f\left(t^2 - \frac{2v}{f}t\right) + 300$$
$$= \tfrac{1}{2}f\left(t^2 - \frac{2v}{f}t + \frac{v^2}{f^2} - \frac{v^2}{f^2}\right) + 300$$
$$= \tfrac{1}{2}f\left(t - \frac{v}{f}\right)^2 + 300 - \frac{v^2}{2f}.$$

Now d is positive for all values of t if $300 - \frac{v^2}{2f} > 0$, or $v^2 < 600f$ and then the man will never catch the bus.

The man will just catch the bus in one minute if $t = \frac{v}{f} = 60$ and $v^2 = 600f$, d being then zero.

In this case, $v^2 = 3600f^2 = 600f$

$$\text{or, } 600f(6f - 1) = 0$$

which gives $f = \frac{1}{6}$ ($\because$ $f \neq 0$)

so that $v = 60f = 10$.

Hence the man runs with a speed of 10 ft./sec. and the acceleration of the bus is then $\frac{1}{6}$ ft./sec^2.

If $v = 8$, we have $300 - \frac{v^2}{2f} = 300 - \frac{8^2}{2 \times \frac{1}{6}} = 108 > 0$.

Hence the man can never catch the bus. In this case, d is least when $t = \frac{v}{f}$ and then

$$d = 300 - \frac{v^2}{2f} = 108.$$

Thus the least distance by which the bus is ahead of the man is 108 ft. = 36 yds.

EXAMPLE 4

Two particles start together and move from the same point A along the same line AB. P has a uniform velocity of 20 ft./sec. and Q a uniform acceleration of 4 ft./sec^2 and no initial velocity. Find when and where they meet again. Before they meet again, find when the distance between the two will be maximum and what would be maximum distance.

Let us suppose that the particles P and Q meet at C after t secs.

$\therefore \quad AC =$ distance described by P in t secs.
$= 20t$

and $AC =$ distance described by Q in t secs.
$= \frac{1}{2} \cdot 4t^2 = 2t^2$.

$\therefore \quad 20t = 2t^2$, or $2t(10 - t) = 0$ so that $t = 0, 10$. $t = 0$ gives the starting time.

$\therefore$ The particles meet after 10 secs at C where $AC = 20.10$ ft. $= 200$ ft.

Let x be the distance between P and Q at any time t before they meet again.

Then $x = 20t - 2t^2 = -2(t^2 - 10t)$
$= -2(t^2 - 10t + 25 - 25)$
$= 50 - 2(t - 5)^2$.

$\therefore$ x will be maximum when $(t - 5)^2$ is minimum i.e. when $t - 5 = 0$ or $t = 5$ secs, and then $x = 50$ ft.

EXAMPLE 5

A particle starting from rest moves with uniform acceleration. At what time will it be in a position to move as much distance, in the next second, as it has already covered ?

Let the required time be t secs. Then the distance d covered by the particle in t secs. is equal to the distance covered in the $(t + 1)$th second.

$$\therefore \quad d = \tfrac{1}{2}ft^2 \qquad (1)$$

$$\text{and} \quad d = \tfrac{1}{2}f\{2(t+1) - 1\}, \qquad (2)$$

f being the uniform acceleration of the particle.

By (1) & (2), $\frac{1}{2}ft^2 = \frac{1}{2}f(2t + 1)$

or, $t^2 - 2t - 1 = 0$

which gives $t = \frac{2 \pm \sqrt{4 + 4}}{2} = 1 \pm \sqrt{2}$.

Since t cannot be negative, the value $t = 1 - \sqrt{2}$ is not taken into account.

$\therefore$ The required time $= (1 + \sqrt{2})$ secs.
$= 2.414$ secs approximately.

EXAMPLE 6

A bullet moving with a velocity of 1600 ft./sec. has its velocity reduced to one-half after penetrating one inch into a target. Assuming the resistance to be uniform, how far will it penetrate before its velocity is destroyed ?

Let f ft./sec^2 be the retardation of the motion of the bullet due to the resistance of the wall, supposed uniform.

By the given condition,

$$800^2 = 1600^2 - 2 \times f \times \frac{1}{12}$$

$$\text{or,} \quad f = 6(1600^2 - 800^2) \qquad (1)$$

Let x ft. be the total distance penetrated through the wall.

Then $0 = 1600^2 - 2fx$

$$\text{or,} \quad 2fx = 1600^2 \qquad (2)$$

Dividing (2) by (1), we get

$$2x = \frac{1600^2}{6(1600^2 - 800^2)}$$

$$\therefore \quad x = \frac{1600^2}{12(1600 + 800)(1600 - 800)}$$

$$= \frac{\overset{2}{\cancel{1600}} \times \overset{2}{\cancel{1600}}}{\underset{\underset{3}{\cancel{6}}}{\cancel{12}} \times \underset{3}{\cancel{2400}} \times \cancel{800}} = \frac{1}{9}$$

$$\therefore \quad x = \frac{1}{9} \text{ ft.} = \frac{4}{3} \text{ inches.}$$

Hence the bullet penetrates $\frac{4}{3}$ inches through the wall before its velocity is destroyed.

EXAMPLE 7

A particle starts from the origin with velocity 4 m./sec. at time $t = 0$ and moves with an acceleration which decreases at a constant rate from 6 m./sec^2 at $t = 0$ to zero at $t = 2$. Find the distance covered in the two seconds.

Let v be the velocity and f be the acceleration of the particle at any time t.

By the given condition $\frac{df}{dt} = -k\,(k > 0)$.

On integration with respect to t, we get

$$f = A - kt \tag{1}$$

where A is the constant of integration.

Since $f = 6$ at $t = 0$, we have from (1), $A = 6$.

$$\therefore \quad \text{(1) becomes } f = \frac{dv}{dt} = 6 - kt. \tag{2}$$

Again, $f = 0$ when $t = 2$.

$\therefore \quad k = 3$ from (2).

Hence (2) becomes $\frac{dv}{dt} = 6 - 3t$

which gives on integration with respect to t, $v = 6t - \frac{3}{2}t^2 + B$, B being a constant.

We are given that $v = 4$ at $t = 0$.

$\therefore \quad B = 4$ and we have

$$v = \frac{dx}{dt} = 6t - \tfrac{3}{2}t^2 + 4 \tag{3}$$

where x is the distance described by the particle in time t.

Integrating (3) again, we get

$$x = 3t^2 - \tfrac{1}{2}t^3 + 4t + C$$

where C is constant.

The initial condition is $x = 0$ at $t = 0$ so that $C = 0$.

$$\therefore \quad x = 3t^2 - \tfrac{1}{2}t^3 + 4t$$

$$\therefore \quad \text{When } t = 2,\ x = 3 \,.\, 2^2 - \tfrac{1}{2} \,.\, 2^3 + 4.2$$
$$= 12 - 4 + 8 = 16$$

$\therefore$ Distance described in the two seconds is 16 metre.

EXAMPLE 8

If $f = -kv$ (where k is a positive constant), and if $x = 0$, $v = u$ when $t = 0$, show that x tends to a limiting value as $t \to \infty$, and that in any interval the decrease in velocity is proportional to the distance covered.

We have $\frac{dv}{dt} = f = -kv\,(k > 0)$.

Separating the variables and integrating, we get

$$\log v = -kt + \log A \quad (A \text{ being a const.})$$
$$\text{or,} \quad v = Ae^{-kt}.$$

Since $v = u$ when $t = 0$, we have $u = A$.

$$\therefore \quad v = \frac{dx}{dt} = ue^{-kt}. \tag{1}$$

Separating the variables and integrating again, we have $x = -\frac{u}{k}e^{-kt} + B$, B being a constant.

Since $x = 0$ at $t = 0$, we get $B = \frac{u}{k}$.

$$\text{Hence} \quad x = \frac{u}{k}\,(1 - e^{-kt}). \tag{2}$$

As $t \to \infty$, $e^{-kt} \to 0$ since $k > 0$, and therefore $x \to \frac{u}{k}$ by (2).

Thus x tends to the limiting value $\frac{u}{k}$ as $t \to \infty$.

Again, the decrease in velocity in time t

$$= u - v = u - ue^{-kt} \quad \text{by (1)}$$
$$= u\,(1 - e^{-kt})$$
$$= kx \quad \text{by (2)}$$
$$\text{i.e.} \propto x.$$

Thus the decrease in velocity in any interval of time is proportional to the distance covered.

Exercise 12.1

1. A body starting from rest moves with an acceleration of 5 cm./sec^2; in what time will it acquire a velocity of 25 cm./sec., and what

distance does it traverse in that time?

2. A body starting from rest and moving with uniform acceleration describes 133 feet in the tenth second. Find its acceleration.

3. Find the distance described by a particle in the 10th second of its motion, if its initial velocity be 60 ft. per second and retardation be 2 ft. per second per second.

4. A particle moving with uniform acceleration passes over 19 ft. in the fourth second and 27 ft. in the sixth second of its motion. Find the initial velocity and acceleration.

5. A point, moving with uniform acceleration, describes 25 ft. in the half second which elapses after the first second of its motion, and 198 feet in the eleventh second of its motion; find the acceleration of the point and its initial velocity.

6. The speed of a train is reduced from 40 miles an hour to 10 miles per hour, whilst it travels a distance of 150 yards; if the retardation be uniform, find how much further it will travel before coming to rest.

7. A bullet fired into a target loses half its velocity after penetrating 3 inches. How much further will it penetrate?

8. A particle, moving in a straight line with uniform acceleration, passes over 25 ft. in the 5th sec. and 33 ft. in the 7th sec. of its motion. Find its initial velocity and acceleration.

9. A particle starting with a given velocity moves for 3 secs. with constant acceleration, during which time it describes 81 ft.; the acceleration then ceases and during the next 3 secs., it describes 72 ft. Find its initial velocity and acceleration.

10. A bullet passes through a wall 9.6 inches thick and its velocity changes from 1200 to 800 ft./sec. thereby. Find the time required by the bullet to pass through the wall, and the velocity when half the wall is penetrated.

11. A train stops at two stations 4 miles apart and takes 8 minutes on the journey from one station to the other. If its motion is first that of uniform acceleration x and then that of uniform retardation y, find the relation between x and y.

12. A train travels from a station X to a station Y in 45 minutes. At a point Z, somewhere between X and Y, it attains its maximum velocity of 45 m.p.h. If it travels with uniform acceleration from X to Z and uniform retardation Z to Y, find the distance between X and Y, it being supposed that the train starts from rest at X and comes to rest at Y.

13. A particle moving along a straight line with uniform acceleration describes 7 ft. during the 5th sec. of its motion, and ultimately comes to rest after sometime. If it describes $\frac{1}{64}$th of the whole distance during the last second of its motion, find how long it was in motion and also its initial velocity.

14. If a, b, c be the spaces described in the pth, qth and rth seconds by a body starting with a given velocity u and moving with uniform acceleration f, show that $a(q-r)+b(r-p)+c(p-q)=0$.

15. Prove that for a particle moving with uniform acceleration f in a straight line, $f = 2\left(\frac{s'}{t'}-\frac{s}{t}\right)\Big/(t+t')$ where s is the space described in t seconds, and s' during the next t' seconds.

16. A train travels from rest at one station to rest at another (in the same straight line) distant d ft. It moves for the first part of the distance with an acceleration of a ft. per sec^2 and for the remainder with a retardation of b ft. per sec^2. Show that it will accomplish the journey in $\sqrt{\left\{\frac{2(a+b)d}{ab}\right\}}$ secs.

17. A point moving under uniform acceleration describes successive equal distances in times t_1, t_2, t_3. Show that $\frac{1}{t_1}-\frac{1}{t_2}+\frac{1}{t_3} = \frac{3}{t_1+t_2+t_3}$.

18. Two particles start at the same instant

from a point A and move in the same direction along a straight line AB. The first has a uniform velocity of 40 ft./sec. while the second starts with an initial velocity of 16 ft./sec. and has a uniform acceleration of 6 ft./sec^2. Find the time that elapses before the two particles meet again.

19. A particle moves with an acceleration which is inversely proportional to its velocity, and starts from the origin with speed u cm./sec. After 3 seconds its speed is $2u$. Find the distance covered in the 3 secs.

20. A point starts from the origin at time $t = 0$ with velocity 8 m./sec. Its acceleration, which is initially $-$ 4 m./sec^2, increases steadily with the time at such a rate that at a certain instant the velocity and the acceleration are both momentarily zero. Show that at this instant the distance covered is $\frac{32}{3}$ m.

21. A point moves so that the graph of v against x is a straight line cutting the x-axis at $x = 100$. At time $t = 0$, $x = 0$ and $v = 50$ cm./sec. Find v and x in terms of t and show that it would take an infinite time for the point to reach $x = 100$.

22. A point starts moving with velocity u at time $t = 0$. Its average velocity in any interval of time starting at $t = 0$ and ending at $t = t$ is $\frac{1}{3}(u + 2v)$, where v is its velocity at time t. Show that its acceleration is proportional to $t^{-\frac{1}{2}}$.

23. A tram runs along a straight track. After starting, the distance travelled by the tram is proportional to the cube of time. During the first minute the tram travelled 90 metres. Find the velocity and acceleration for $t = 0$ and $t = 5$ secs.

24. The landing speed of an aircraft is 100 km./h. Determine its deceleration at landing on the track of $l = 100$ m. length if the deceleration is constant.

25. A particle is moving in a straight line, and is observed to be at a distance a from a marked point initially, to be at a distance b after an interval of n sec., to be at a distance c after $2n$ sec., and at a distance d after $3n$ sec. If the acceleration be uniform, prove that $d - a = 3(c - b)$ and that the acceleration is $\dfrac{c+a-2b}{n^2}$.

Find also the initial velocity.

12.4 Falling bodies

It is well known that the speed of a heavy body falling towards the earth, increases as it falls or that it moves with an acceleration.

Natural philosophers of the sixteenth century and earlier believed that a body falls towards the earth with a speed proportional to its weight so that two bodies of different weights would not fall in the same manner. It was Galileo (1564-1642) who carried out an experiment, for the first time, on the motion of falling bodies. He established that

> a body falling freely (that is, air resistance is neglected) towards the earth's surface, moves in a straight line and its distance below the starting point is proportional to the square of the time during which it has been falling.

Thus if the body falls through a distance x in time t, we have

$x = kt^2$, k being a constant.

Hence the velocity at time t is

$$\frac{dx}{dt} = 2kt$$

and the acceleration then is

$$\frac{d^2x}{dt^2} = 2k,$$

both being directed vertically downwards. Thus all bodies falling freely near the earth's surface at any given place have the same acceleration acting vertically downwards.

This acceleration is known as **acceleration due to gravity** and is denoted by g. The magnitude of g varies with position on the earth's surface and also with elevation above the sea level. Unless otherwise stated, we shall take $g = 32$ ft./sec^2 or 981 cm/sec^2.

The motion of any falling body being uniformly accelerated and rectilinear, all the formulae established in Section 12.2 hold good here also.

12.4(*a*) *Vertical motion of a particle falling freely under gravity*

Let a particle fall (from rest) freely under gravity from a point O at a height $h = OA$ above the earth's surface. The point O is taken as the origin and the downward vertical line through O is taken as the positive x-axis.

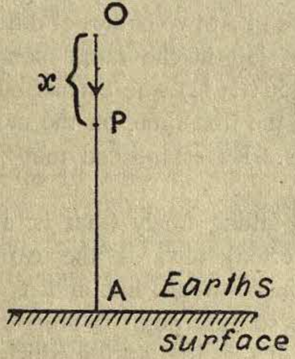


Fig. 12.4

Let the particle attain a velocity v when it has fallen through a distance $x(= OP)$ in time t. Putting $u = 0$ and $f = g$ in the formulae of Section 12.2, we have

$$v = gt, \tag{1}$$

$$x = \tfrac{1}{2}gt^2, \tag{2}$$

$$v^2 = 2gx. \tag{3}$$

Putting $x = h$ in (2) and (3), we get

T = time of falling to the earth's surface

$$= \sqrt{\frac{2h}{g}} \tag{4}$$

and V = velocity with which the particle falls to the earth $= +\sqrt{2gh}$. (5)

[The positive sign of the square root is taken because v is directed in the positive x-direction.]

12.4(*b*) *Motion of a particle moving vertically upwards*

It is our common experience that when a body is projected vertically upwards with some velocity, its speed gradually decreases, it comes to rest (when the greatest height is attained) for an instant and then falls with the acceleration due to gravity. During the upward motion, it has the downward acceleration due to gravity, or a retardation g.

Let us suppose that a particle is projected with an initial velocity u, vertically upwards from a point O on the earth's surface. The upward vertical line through O is taken as the positive x-axis, O being the origin. The particle experiences a retardation or negative acceleration g. We shall assume, for simplicity, that g is constant, i.e. we shall neglect the variation of g with height, and also the resistance of the atmosphere.

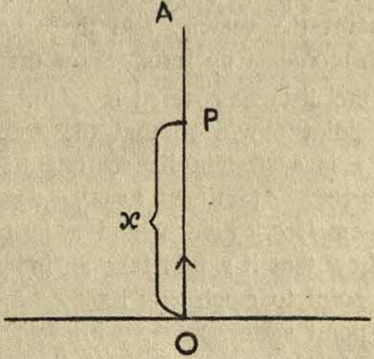


Fig. 12.5

If v be the velocity attained by the particle at P when it has gone up through a height x in time t, the equations relating to motion in a straight line with constant acceleration (Section 12.2) give

$$v = u - gt, \tag{1}$$

$$x = ut - \tfrac{1}{2}gt^2, \tag{2}$$

$$v^2 = u^2 - 2gx. \tag{3}$$

It is clear from (1) that (i) $v > 0$ as long as $u - gt > 0$ or $t < \frac{u}{g}$ and (ii) $v = 0$ when $t =$

$\frac{u}{g}$. Again (3) gives $x = \frac{u^2}{2g}$ when $v = 0$. Thus, if $H(= OA)$ be the greatest height attained in time T_1, we have

$$T_1 = \text{time of flight upwards} = \frac{u}{g} \qquad (4)$$

and $H = \dfrac{u^2}{2g}$. (5)

We find from (2) that $x = 0$ when $t = 0$ or $t = \frac{2u}{g}$. Here $t = 0$ is the time of start and $t = \frac{2u}{g}$ gives the time after which the particle comes back to the starting point O. Putting $t = \frac{2u}{g}$ in (1), we have $v = -u$. Hence

T = total time of flight (from A to O and back to O) $= \dfrac{2u}{g}$ (6)

and V = velocity with which the particle comes back to the earth $= -u$. (7)

REMARKS

1. We find from (1) that $v < 0$ when $t > \frac{u}{g}$ and that v increases numerically as t increases. This shows that, after reaching the highest point, the particle begins to descend and its speed increases.

Putting $x = 0$ in (3), we have $v = \pm u$.

The + sign gives the velocity on starting and the − sign the velocity on returning to the point of projection.

From (4) and (6), we find $T = 2T_1$.

Hence the *time of flight* is twice the *time of rise*.

Also, T_2 = time of fall (from A to O)

$$= T - T_1 = \frac{2u}{g} - \frac{u}{g} = \frac{u}{g}.$$

Hence *time of rise = time of fall*.

2. Equation (2) may be rearranged as

$$gt^2 - 2ut + 2x = 0.$$

This being a quadratic equation in t, it has two roots, namely

$$t = \frac{2u \pm \sqrt{4u^2 - 4g.2x}}{2g} = \frac{u \pm \sqrt{u^2 - 2gx}}{g}$$

$$= \frac{u}{g} \pm \frac{\sqrt{u^2 - 2gx}}{g}$$

These roots are real only if $u^2 \geqslant 2gx$. Let us write $t_1 = \frac{u}{g} - \frac{\sqrt{u^2 - 2gx}}{g}$, $t_2 = \frac{u}{g} + \frac{\sqrt{u^2 - 2gx}}{g}$.

Here $\frac{u}{g}$ being the time to attain the greatest height, $t_1 \left(< \frac{u}{g}\right)$ represents the time in going up to some point P at a height x above the ground and $t_2 \left(> \frac{u}{g}\right)$ represents the time after which the particle passes through P on the way down.

It is to be noted further that t_1, t_2 become imaginary when $x > \frac{u^2}{2g}$, i.e. x is greater than the greatest height attained by the particle.

Again, we have from (3)

$$v = \pm \sqrt{u^2 - 2gx}$$

Here also v is real only if $x \leqslant \frac{u^2}{2g}$. The + sign represents the velocity at P on the way up and the − sign that on the way down.

3. In order to determine the time taken to reach a point *below* the point of projection, it is not necessary to find the time up and down to the point of projection and then the time to the point below. We simply substitute *the distance below the point of projection* for x in (2), giving it a negative sign. This is illustrated in Example 2 worked out below.

12.5 Illustrative examples

EXAMPLE 1

A body falls from rest. Find (i) how far it will fall in 10 secs. ; (ii) how long it takes to fall 100 ft. and (iii) its velocity after falling 100 ft.

(i) d = distance fallen through in 10 secs.

$$= \frac{1}{2}gt^2 = \frac{1}{\cancel{2}} \times \overset{16}{\cancel{32}} \times 100 \text{ft.} = 1600 \text{ ft.}$$

(ii) Let t be the time to fall 100 ft.

$$\text{Then } 100 = \frac{1}{2} \times 32 \times t^2 = 16t^2$$

$$\text{or, } t^2 = \tfrac{25}{4}. \therefore \ t = \pm \tfrac{5}{2}$$

$\therefore$ Required time $= \frac{5}{2}$ secs. The negative value is obviously inadmissible.

(iii) Let v be the velocity of the body after it has fallen through a distance of 100 ft.

$$\text{Then } v^2 = 2gx = 2 \times 32 \times 100 = 6400$$

$$\therefore \ v = 80$$

$\therefore$ The velocity of the body is 80 ft./sec. after it falls through 100 ft.

EXAMPLE 2

A ball is thrown vertically upwards with a speed of 128 ft. per sec. Find where it is after 5 secs., and the total distance it has actually travelled. If it falls past the point of projection into a well of depth 120 ft., find when it strikes the bottom.

Time of rise to the greatest height

$$= \frac{u}{g} = \frac{128}{32} \text{ secs.} = 4 \text{ secs.}$$

The ball then descends with initial velocity zero for $(5 - 4)$ sec. or 1 sec. to reach the required point.

Now, distance fallen through in 1 sec.

$$= \tfrac{1}{2}gt^2 = \tfrac{1}{2} \times 32 \times 1^2 \text{ ft.} = 16 \text{ ft.}$$

Also, the greatest height attained by the ball

$$= \frac{u^2}{2g} = \frac{128 \times \overset{2}{\overset{\cancel{4}}{\cancel{128}}}}{\cancel{2} \times \cancel{32}} \text{ ft.} = 256 \text{ ft.}$$

$\therefore$ The ball has actually travelled, in 5 secs., a distance $(256 + 16)$ ft. or 272 ft., and then it is at height of $(256 - 16)$ ft. or 240 ft. above the point of projection.

Let t be the time after which it strikes the bottom of the well.

Using $x = ut - \frac{1}{2}gt^2$

and putting $x = -120$, $u = 128$, $g = 32$, we get

$$-120 = 128t - 16t^2$$

$$\text{or, } 16t^2 - 128t - 120 = 0$$

$$\text{or, } 2t^2 - 16t - 15 = 0$$

$$\therefore \ t = \frac{16 \pm \sqrt{256 + 120}}{4} = \frac{16 \pm 19.38}{4}$$

approx.

$$= 8.8 \text{ or } -0.8$$

Since t cannot be negative, the body strikes the bottom of the well after nearly 8.8 secs.

EXAMPLE 3

A particle is projected vertically upwards with a velocity of u ft./sec., and after t secs. another particle is projected upwards with the same velocity from the same point. Prove that the particles will meet after a lapse of $\left(\frac{t}{2} + \frac{u}{g}\right)$ secs. from the instant of projection of the first particle. Find also the height of their meeting point.

Let the two particles meet at a height h ft. after a lapse of t' secs. from the instant of projection of the first particle.

$$\text{Then } h = ut' - \tfrac{1}{2}gt'^2 \qquad (1)$$

$$\text{and } h = u(t' - t) - \tfrac{1}{2}g(t' - t)^2 \qquad (2)$$

$$\text{By (1) \& (2), } \cancel{ut'} - \cancel{\tfrac{1}{2}gt'^2} = \cancel{ut'} - ut - \cancel{\tfrac{1}{2}gt'^2} - \tfrac{1}{2}gt^2 + gt't$$

$$\text{or, } gt't = t(u + \tfrac{1}{2}gt)$$

$$\therefore \ t' = \frac{u}{g} + \frac{t}{2}$$

Putting this value of t' in (1), we get

$$h = t'\left(u - \frac{gt'}{2}\right)$$

$$= \left(\frac{u}{g} + \frac{t}{2}\right)\left(u - \frac{u}{2} - \frac{gt}{4}\right)$$

$$= \frac{(2u + gt)}{2g} \cdot \frac{2u - gt}{4} = \frac{4u^2 - g^2t^2}{2g}$$

Hence the particles meet at a height of $\frac{4u^2 - g^2t^2}{8g}$ ft. after a lapse of $\left(\frac{u}{g} + \frac{t}{2}\right)$ secs.

from the instant of projection of the first particle.

Exercise 12.2

1. A particle is projected vertically upwards from a point O with velocity 60 ft./sec. At what times will it be (i) 44 ft. above, (ii) 16 ft. below O?

2. A particle is projected vertically upwards with a velocity of 40 ft./sec. Find (i) when its velocity will be 55 ft./sec. and (ii) when it will be 25 ft. above the point of projection.

3. A body is projected vertically upwards with a velocity of 80 ft./sec. Find how long it takes to return to the point of projection.

4. A body is projected vertically upwards with a velocity of 96 ft./sec. Find (i) how long it takes to reach its highest point; (ii) the distance it ascends during the 3rd sec. of its motion.

5. A particle falling from rest moves through 240 ft. during the last second before its strikes the ground. Find the height from which it falls.

6. A body falls freely from the top of a tower, and during the last second it falls $\frac{9}{25}$th of the whole distance. Find the height of the tower.

7. A body is projected vertically downwards from the top of a tower with a velocity of 40 ft./sec., and it takes 3 secs. to reach the ground. What is the height of the tower?

8. The greatest height attained by a particle projected vertically upwards is 225 ft.; find how soon after projection the particle will be at a height of 176 ft.

9. A body which fell from rest was observed to fall through 256 ft. in 2 secs. For how long had it been falling before it was observed?

10. A particle is projected upwards, from the ground, and after sometime it is seen at a height of 21 ft. falling downwards with a velocity of 16 ft./sec. How long before this was it moving upwards through the same point and what was its velocity then? Find also the time from the start to the highest point.

11. A particle is projected vertically upwards. Prove that it will be at $\frac{3}{4}$ of its greatest height at times which are in the ratio 1 : 3.

12. If a stone falls past a window 8 ft. high in half a sec., find the height from which it fell.

13. A particle is projected vertically upwards with speed 128 ft./sec. from a point O; it passes successively through two horizontal screens at heights of 10 ft. and 20 ft. respectively above O. If the speed of the particle be halved on passing through a screen, find the greatest height above O that the particle attains. If the speed is also halved on passing through such screen on the downward journey, find its speed on coming back to O.

14. A sandbag is dropped from an airship which is 1400 ft. above the ground and is rising vertically at the rate of 20 ft./sec. Find the time the sandbag takes to reach the ground.

15. A stone dropped into a well reaches the water with a velocity of 80 ft./sec. and the sound of its striking the water is heard in $2\frac{7}{12}$ secs. after it is let fall. Find the velocity of sound.

16. A stone is dropped into a well, and the sound of the splash is heard in $7\frac{7}{10}$ secs. If the velocity of sound be 1120 ft./sec., find the depth of the well.

17. A stone is dropped from a height of 50 ft. above the ground. At the same time, a ball is projected upwards from the ground with a velocity of 40 ft./sec. in the same vertical line. Show that they will meet midway, and find the time of meeting.

18. A body is projected vertically upwards with an initial velocity of 80 ft./sec. Find the height to which the body rises.

Another body is projected vertically upwards 2 secs. after the first with an initial velocity of 64 ft./sec. Find when the two bodies are at the same height, and find also the velocity of each body at this instant.

19. A particle is projected vertically upwards from a point O with an initial speed of 80 ft./sec. During what interval of time is the particle more than 96 ft. above A ?

With what velocity must a second particle be projected vertically upwards from A at the beginning of that interval in order that the particle should meet at the end of that interval ?

20. A body is dropped from the top of a tower 144 ft. high, and just at the moment, another body is projected vertically upwards along the same line from the foot of the tower with a velocity just sufficient to carry it to 256 ft. What time elapses before they meet ?

21. A particle thrown vertically upwards takes t secs. to rise to a height h and t' secs. is the time taken to reach the ground again. Show that $h = \frac{1}{2}gtt'$.

22. A particle projected vertically upwards is in the same position after $n - 1$ sec. as it is after n sec. Find the starting velocity and the greatest height above the starting point.

23. A rocket ascending vertically from the ground with an initial velocity of $\sqrt{2gy}$ ft./sec. explodes when it reaches the greatest height and the interval between the sound reaching the place of starting and a place x ft. distant from it is $\frac{1}{n}$ of a second. Show that the velocity of the sound is $n(\sqrt{x^2 + y^2} - y)$ ft./sec.

24. A particle is projected vertically upwards with a velocity of u ft./sec., and t secs. afterwards, another particle is projected vertically upwards with the same initial velocity. Prove that their velocities when meeting will be each $\frac{1}{2}gt$.

25. A balloon rises from rest on the ground with constant acceleration $\frac{1}{8}g$. A stone is dropped when the balloon has risen to a height H ft. Find the time taken by the stone to reach the ground. If a second stone is dropped t secs. after the first one, find the distance between the stones t secs. after the second stone is dropped, assuming that the first stone has not reached the ground at that time.

13 Newton's Laws of Motion

We have so far discussed motions of particles without considering the causes of such motions. In the present chapter we intend to examine the causes of motion of a body. The causes of motion of a body are embodied in three basic laws enunciated by Sir Issac Newton and published in his famous book *Principia* in 1686. These laws are known as **the laws of motion** and they admit of no strict proof, experimental or otherwise. They are the basic postulates or assumptions on which the science of Newtonian Dynamics is based, and like other scientific laws, they are justified only in so far as they lead to results in agreement with the observations made. The results obtained from these laws and the predictions made in Dynamical Astronomy are found to agree with the observations remarkably well, and thus provide excellent confirmation of these fundamental laws.

13.1 Newton's Laws of Motion

First Law: Every body continues to remain in its state of rest or of uniform motion along a straight line unless it is compelled to change that state by external impressed forces.

Second Law: The rate of change of momentum of a body is proportional to the resultant external force acting on it and takes place in a direction in which the force acts.

Third Law: To every action there is always an equal and opposite reaction.

13.2 The first law of motion

The first law gives essentially the definition of a force. It states that a body cannot change its state of rest or of uniform motion (rectilinear) by its own effort and for any kind of change of its existing state, an external agent is necessary ; this is what we call a **force.** Hence, **a force is that which produces or tends to produce a change in the existing state of rest of a body, or of its uniform motion in a straight line.** If the body be not compelled to change its existing state by a force, it continues to remain in its existing state. This property of a body to retain its existing state is known as **inertia.**

13.3 The second law of motion

We have found that the first law of motion gives the qualitative definition of a force. The second law helps us to obtain the quantitative measurement of a force. We should first define the **momentum** of a moving body.

The momentum of a body is the product of its mass m (supposed to be a constant) and its velocity v at any time t.

Now, we have from the first part of the second law,

$$P \propto \frac{d}{dt}\,(mv)$$

where P is the force acting on the body at time t ;

i.e. $P \propto m\,\dfrac{dv}{dt}$ (supposing m to be a constant);

i.e. $P \propto mf$

where f is the acceleration of the body at time t ;

i.e. $P = Kmf$, K being a constant to be determined.

To determine K, let us introduce the idea of **a unit force.**

A unit force produces a unit acceleration acting on a unit mass. Thus $P = 1$ when $m = 1$ and $f = 1$. Hence $K = 1$ and we have

$$P = mf.$$

Here P and f being both vector quantities, the rate of change of momentum (i.e. mf) takes place in the direction in which the force P acts.

$$\therefore\quad P = m\,\frac{dv}{dt} = mv\,\frac{dv}{ds}\,.$$

This fact leads us to an important physical concept which is as follows:

> If a number of forces act simultaneously in different directions on a body having a certain initial velocity, each force produces an acceleration in the direction in which it acts, irrespective of the influences of the other.

This is known as the **Principle of Physical Independence of Forces.**

REMARKS

The second law coupled with the parallelogram law of accelerations leads us to the **parallelogram law of forces.**

Let the forces P_1 and P_2 acting on a body of mass m produce in it accelerations f_1 and f_2 respectively.

Then $P_1 = mf_1$ and $P_2 = mf_2$. The resultant R of P_1 and P_2 is the resultant mf of mf_1 and mf_2. The resultant of mf_1 and mf_2 can be obtained by using the parallelogram law of accelerations as shown diagramatically.

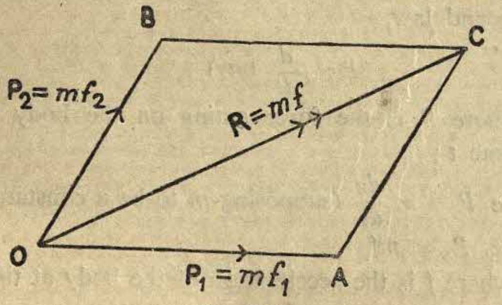


Fig. 13.1

We thus conclude:

> If a particle be subjected to two forces represented in magnitude and direction by two given straight lines drawn from a given angular point, the forces are equivalent to a single force represented in magnitude and direction by the diagonal through that angular point of the parallelogram whose adjacent sides are the straight lines representing the forces.

This is known as the **Parallelogram Law of Forces.**

13.4 Absolute units of force

The unit of a force which, acting on a unit mass, produces a unit acceleration in it is called the absolute unit of force.

The absolute unit of force in the F.P.S. system is called **poundal** which is defined as follows:

> The force which acting on a mass of one pound produces in it an acceleration of 1 ft./sec.^{2} is called a poundal.

The unit of force in the C.G.S. system is known as **dyne** which is defined as:

> A dyne is a force which acting on a mass of 1 gram produces in it an acceleration of 1 cm./sec^2.

We now deduce the relation between a poundal and a dyne.

Since 1 ft. = 30.4 cms (approx.)
and 1 lb. = 453.6 gms,

$$1 \text{ poundal} = 30.4 \times 453.6 \text{ dynes}$$
$$= 13800 \text{ dynes (approx.)}$$

13.5 Gravitational units of force

The force with which a unit mass is attracted by the earth (or gravity) is denoted by g. Hence the force with which a body of mass m is attracted by the earth is mg; it is known as the weight of the body and is denoted by W so that $W = mg$. It follows that g represents an acceleration and it is called *acceleration due to gravity.*

We have $g = 32$ ft./sec^2 (approx.) in the F.P.S. system and $g = 981$ cms./sec.^{2} (approx.) in the C.G.S. system.

Hence weight of a mass of 1 lb. = 32 poundals (approx.)
and weight of a mass of 1 gm. = 981 dynes (approx.)

The *weight of a mass of* 1 *lb.* is often termed *as* 1 *lb.-weight* and the *weight of a mass of* 1 *gm.* as 1 *gm.-weight.* Here lb.-weight and gm.-weight are called **gravitational units of force.** Thus the magnitude of a force in gravitational units may be obtained by dividing its magnitude (i) in poundals by 32 in the F.P.S. system and (ii) in dynes by 981 in the C.G.S. system.

13.6 The third law of motion

The third law, known as the **Law of Action and Reaction,** determines the character of mechanical interaction between material bodies. In the case of two particles, the law states: *Two particles exert on each other forces equal in magnitude and acting in opposite directions along the straight line connecting the two particles.* The law implies that whenever a body exerts a force on another body, the latter exerts at once an equal and opposite force on the former. The force exerted by the former body is termed as **action** and that exerted by the latter is called **reaction.** This law of action and reaction holds whether the first body is in contact with the second body or they attract or repulse each other from some distance. When a book is kept on a table, the weight of the book **acts** on the table and the table **exerts** on the book a force (reaction) which is equal and opposite to the weight of the book.

13.7 To find out the pressure of a body resting on a moving horizontal plane

Let a body of mass m rest on a horizontal plane which is moving vertically upwards or downwards.

CASE I: *The plane is moving vertically upwards with an acceleration f.*

By the third law of motion, the pressure exerted by the body on the plane is equal and opposite to the reaction of the plane on it. Let R be the reaction of the plane on the body. The downward force of gravity on the body is its weight mg. Hence the resultant upward force causing the motion is $R - mg$. By the second law of motion, we now have

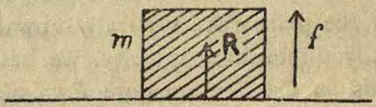


Fig. 13.2

$$R - mg = mf$$

$$\text{or,}\quad R = m(g + f). \qquad (1)$$

Hence the pressure exerted by the body on the plane is $F = m(g + f)$ downwards when the plane is moving vertically upwards with an acceleration f.

CASE II: *The plane is moving vertically downwards with an acceleration f.*

The resultant force on the body being now downwards, the reaction R is less than the weight mg. The resultant downward force which causes the body to descend vertically with the acceleration f is $mg - R$. By the second law of motion,

$$mg - R = mf$$

$$\text{or,}\quad R = m(g - f). \qquad (2)$$

Hence the pressure exerted by the body on the plane is $F = m(g - f)$ downwards when the plane is moving vertically downwards with an acceleration f.

COROLLARY

When the plane is at rest or moves upwards or downwards (vertically) with a uniform velocity, we have $f = 0$ and therefore, $R = mg$.

Hence the pressure exerted by the body on the plane is just equal to the weight of the body.

REMARKS

1. Formulae (1) and (2) fully comply with our everyday experience that (i) a man on a rising lift feels himself heavier than his actual weight and (ii) a man on a descending lift feels himself lighter than his actual weight.

2. If the plane rises vertically upwards with gradually diminishing velocity, we have $f < 0$ upwards so that the pressure $F = m(g + f)$

$$< mg.$$

Similarly, when the plane descends vertically downwards with a gradually diminishing velocity, we have $f < 0$ downwards and then

$$F = m(g - f) > mg.$$

13.8 Illustrative examples

EXAMPLE 1

A force equal to the weight of 1000 grammes acts on a mass of 200 grammes for half a minute. Find the velocity acquired by the mass.

Here the force acting on the mass = 1000 g dynes where g = 981 cm/sec.^{2}

$\therefore$ The acceleration produced in the mass is

$$f = \frac{1000\text{g}}{200}\ \text{cm/sec}^2 = 5\text{g cm/sec.}^2$$

$\therefore$ The velocity acquired by the mass after half a minute $= ft = 5 \times 981 \times 30$ cms./sec.

$$= 147150 \text{ cm/sec.}$$

EXAMPLE 2

A mass of 4 lbs. falls 200 ft. from rest and is then brought to rest by penetrating 2 feet into some mud. Find the average thrust of the mud on it.

Let v be the velocity of the mass just before penetrating into the mud.

Then $v^2 = 2gx = 2 \times 32 \times 200 = 12800$.

Let the average thrust of the mud on the mass be R poundals which acts upwards. The weight of the mass is $4g$ poundals = 128 poundals and it acts downwards.

Now the acceleration opposite to the direction of motion is $\dfrac{R - 4g}{4} = \dfrac{R - 128}{4}$ ft./sec.^{2}

$$\therefore\ 0 = 12800 - \cancel{2} \times \frac{R - 128}{\underset{2}{\cancel{4}}} \times \cancel{2}$$

or, $R = (12800 + 128)$ poundals = 12928 poundals

$\dfrac{12928}{32}$ lbs. wt. = 404 lbs. wt.

EXAMPLE 3

A train runs from rest for 1 mile down an incline of 1 in 100. If the resistance be equal to 8 lbs. wt. per ton, how far will the train be carried along the horizontal level at the foot of the incline ?

Let m be the mass of the train and α be the inclination of the incline with the horizon. Then $\sin\alpha = \dfrac{1}{100}$.

Since the train is moving down the incline, the component of its weight down the incline, i.e. $mg \sin\alpha$, helps its motion while the resistance due to friction, etc. is a retarding force.

$\therefore$ The force helping the motion on the incline

$$\frac{m \times 32}{100} \text{ poundals} = \frac{8m}{25} \text{ poundals}$$

and the retarding force

$$\frac{8m}{2240}\text{ g poundals} = \frac{8m \times \overset{\overset{4}{\cancel{8}}}{\cancel{32}}}{\underset{\underset{\underset{35}{\cancel{70}}}{\cancel{280}}}{\cancel{2240}}} \text{ poundals}$$

$$= \frac{4m}{35} \text{ poundals.}$$

∴ The total force causing the motion

$$\left(\frac{8m}{25}-\frac{4m}{35}\right) \text{ or } \frac{36m}{175} \text{ poundals.}$$

∴ The acceleration of the train on the incline is $f=\frac{36}{175}$ ft/sec². $\left[\because f=\frac{P}{m}\right]$

∴ The velocity v with which it reaches the foot of the incline is given by

$$v^2=2fx=\frac{2\times 36\times 1760\times 3}{175}$$

$[\because\ x = 1 \text{ mile} = 1760 \times 3 \text{ ft.}]$

When the train moves on level road, there is only the retarding force.

∴ If d be the distance covered on the horizontal plane, then

$$0=v^3-2gd=\frac{2\times 36\times 1760\times 3}{175}-\frac{2\times 4\times d}{35}$$

$$\therefore\ d=\frac{\cancel{2}\times\overset{9}{\cancel{36}}\times 1760\times 3\times\overset{7}{\cancel{35}}}{\underset{\underset{5}{\cancel{35}}}{\cancel{175}}\times\cancel{2}\times\cancel{4}} \text{ ft.} = 1\tfrac{4}{5} \text{ miles.}$$

EXAMPLE 4

A man weighing 12 stones is descending in a lift with acceleration 8 ft./sec². Find the thrust of his feet on the lift. Calculate the same when he is ascending with the same acceleration. What would happen to this thrust if the chain of the lift broke (i) during descent, (ii) during ascent ?

Let R be the thrust of the man on the lift which is descending. Here R acts downwards. The reaction of the floor of the lift on the man is also R acting upwards.

$$\therefore\ R=m(g-f)$$

$$= 12 \times 14\ (32 - 8) \text{ poundals}$$

[1 stone = 14 lbs.]

$$=\frac{\cancel{12}\times\cancel{14}\times\overset{3}{\cancel{24}}}{\cancel{14}\times\underset{\cancel{4}}{\cancel{32}}} \text{ stones wt.}$$

$$= 9 \text{ stones wt.}$$

Hence the thrust of the man on the descending lift = 9 stones wt.

Similarly, if R_1 be the thrust of the man on the lift when it is ascending with the same acceleration, we have

$$R_1 = m(g+f) = 12 \times 14\,(32+8) \text{ poundals}$$

$$=\frac{\overset{3}{\cancel{12}}\times\cancel{14}\times\overset{5}{\cancel{40}}}{\cancel{14}\times\underset{\cancel{4}}{\cancel{32}}} \text{ stones wt.}$$

$$= 15 \text{ stones wt.}$$

Hence the thrust of the man on the ascending lift = 15 stones wt.

As soon as the chain of the lift breaks, the lift moves as a free body whose acceleration is g acting downwards.

$$\therefore\ 12 \times 14 \times 32 - R = 12 \times 14 \times 32$$

$$[\because\ mg - R = mg]$$

$\therefore\ R = 0$

Similarly, $R_1 = 0$

Exercise 13.1

1. Find the force expressed (i) in poundals, (ii) in terms of the weight of a pound, that will produce in a mass of 20 pounds an acceleration of 10 ft./sec².

2. A constant force acts upon a mass of 8 lbs. during 4 seconds from rest and then ceases ; in the next 4 seconds it is found that the mass describes 64 feet. Find the magnitude of the force.

3. A mass of 10 pounds is placed on a smooth horizontal plane and is acted on by a force equal to the weight of 3 pounds ; find the distance described by it in 10 seconds.

4. A mass of m pounds, being acted on by a constant force of P poundals for t seconds, moves a distance x ft. from rest. Show that the velocity acquired by the mass is $(2Px/m)^{\frac{1}{2}}$.

5. A heavy truck of mass 16 tons is standing at rest on a smooth line of rails. A horse

now pulls at it steadily in the direction of the line of rails with a force equal to the weight of 1 cwt. How far will it move in 1 minute?

6. A body of mass 3 lbs. is falling under gravity at the rate of 100 ft./sec. What is the uniform force that will stop it (i) in 2 secs., (ii) in 2 ft.?

7. A particle of mass 20 lbs. falls 25 ft. from rest and is then brought to rest by penetrating 6 inches into some mud. Find the average thrust of the mud on the particle.

8. A train whose mass is 300 tons moves at the rate of 60 m.p.h.; after steam is shut off it is brought to rest by the brakes in 50 yards. Find the force exerted on the train, assuming it to be uniform.

9. A force equal to the weight of 10 gms. acts on a mass of 27 gms. for 1 second. Find the velocity of the mass and the distance it has travelled over. At the end of the first second the force ceases to act. How far will the body travel in the next minute?

10. A railway trian whose mass is 100 tons, moving at the rate of 60 m.p.h. in a straight line is brought to rest in 10 secs. by the application of a uniform force. Find how far the train moves during the time for which the force is applied, and calculate the magnitude of the force.

11. A cannon-ball of mass 1000 gms. is discharged with a velocity of 45000 cms./sec. from a cannon the length of whose barrel is 200 cms. Show that the mean force exerted on the ball during the explosion is 5.0625×10^9 dynes.

12. A bullet weighing half-an-ounce leaves the muzzle of a rifle-barrel 2 ft. long with a velocity of 2000 ft./sec. Find the force acting on the bullet in the barrel, assuming it to be uniform; and also the time taken by the bullet to traverse the barrel.

13. A bullet moving at the rate of 200 ft./sec. is fired into a trunk of wood into which it penetrates 9 inches; if a bullet moving with the same velocity were fired into a similar piece of wood 5 inches thick, with what velocity would it emerge, supposing the resistance to be uniform?

14. A shot of mass 100 lbs. moving at the rate of 1600 ft./sec. strikes a fixed target. How far will the short penetrate the target, assuming that it offers an average resistance of the weight of 12000 tons?

15. A train travelling on a level road at the rate of 15 m.p.h. comes to the foot of an incline of 1 in 160 and steam is then turned off. How far will the train go up the incline before it comes to rest, if the resistance due to friction etc. be 14 lbs. wt. per ton?

16. A railway train exclusive of engine weighs 435 tons, and starting along a level line from rest attains a speed of 40 m.p.h. in 7 minutes. Calculate the average pull between the engine and the train, taking the resistance to be 15 lbs. wt. per ton.

17. A scale pan, on which rests a mass of 50 gms., is drawn upwards with a constant acceleration, and the reaction between the mass and the pan is found to be 50,000 dynes. Find the acceleration of the scale pan.

18. A thief jumps off the terrace of a building with a heavy suitcase on his head, and falls vertically. What would be the pressure of the suitcase on his head while he is falling?

19. A man of mass 10 stone stands on a lift which moves with a uniform acceleration of 12 ft./sec^2. Find the reaction of the floor when the lift is (i) ascending, (ii) descending.

20. A thin glass plate can just support a weight of 27 lbs. A body is placed on it and the plate is raised with the body on it with a gradually increasing acceleration. It is found that the plate breaks when the acceleration is 4 ft./sec^2. Find the mass of the body.

21. A mass of 10 lbs. rests on a horizontal plane which is made to ascend (i) with a constant velocity of 5 ft./sec.; (ii) with a constant

acceleration of 5 ft./sec^2. Find in each case the reaction of the plane.

22. In a lift, accelerated upwards at a certain rate, a spring balance indicates a mass to have a weight of 10 lbs. When the lift is accelerated downwards at twice the rate, the mass appears to be 7 lb. in weight. Find the actual weight of the mass, and the upward acceleration of the lift.

23. A body whose true weight was 13 oz. appeared to weigh 12 oz. when weighed by means of a spring balance in a moving lift. What was the acceleration of the lift at the instant of weighing ?

14 Simple Harmonic Motion

14.1 Equation of motion of a particle moving in a straight line

Fig. 14.1

Let a particle of mass m move along a straight line OX starting from O (either from rest or with some initial velocity) under a given force F along OX. Let $x\,(= OP)$ be the displacement and v the velocity of the particle at any time t.

By Newton's Second Law of motion, the equation of motion of the particle is

$$F = m\frac{d^2x}{dt^2} = m\frac{dv}{dt} = mv\frac{dv}{dx} = \frac{m}{2}\frac{dv^2}{dx}\,. \quad \text{(A)}$$

We shall now apply the equation of motion to investigate what is called a Simple Harmonic Motion. We begin with two problems.

14.2 Problem I

A particle starts from rest at A in a straight line OA and moves with an acceleration which is always directed towards O and varies as the distance from O. Discuss the motion.

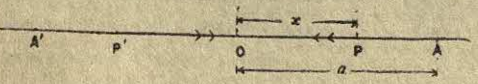


Fig. 14.2

Let P be the position of the particle at any time t from start at A, where $OP = x$ and $OA = a$.

The equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -\mu x \left(x \gtrless 0,\ \mu > 0\right)\,. \quad (1)$$

[For $x > 0$, $\frac{d^2x}{dt^2}$ = acceleration in the x-increasing direction OA

$= -$ (acceleration towards O)

$= -(\mu . OP) \quad = -\mu x.$

For $x < 0$, $\frac{d^2x}{dt^2}$ = acceleration in the direction $P'A$

$= \mu . P'O = \mu(-x) = -\mu x.$]

Equation (1) can be written as

$$\frac{1}{2}\frac{dv^2}{dx} = -\mu x \quad (2)$$

where $v = \frac{dx}{dt}$ denotes the velocity at P.

Since the particle starts from rest at A, we have the initial conditions

$$v = \dot{x} = 0 \text{ when } x = a \text{ at } t = 0. \quad (3)$$

Integrating (2), $v^2 = -\mu x^2 + C$.

Using (3), we have $C = \mu a^2$.

$$\therefore \quad v^2 = \mu\,(a^2 - x^2) \quad (4)$$

and $v = \frac{dx}{dt} = -\sqrt{\mu}\,\sqrt{a^2 - x^2}$. $\quad (5)$

[The '—ive' sign is taken for the square root because the velocity is clearly negative so long as OP is positive and P is moving towards O.]

Integrating (5), $-\int \frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\mu}\int dt + K.$

Let us put $x = a\cos\theta$ so that $dx = -a\sin\theta\, d\theta$.

Then $\sqrt{\mu}\, t + K = \int \frac{a \sin\theta\, d\theta}{a \sin\theta} = \theta$

$$= \cos^{-1}\left(\frac{x}{a}\right).$$

Applying (3), $K = \cos^{-1}(1) = 0$.

$$\therefore \quad \sqrt{\mu}\, t = \cos^{-1}\left(\frac{x}{a}\right)$$

$$\text{or,} \quad x = a \cos \sqrt{\mu}\, t \tag{6}$$

$$\text{Then} \quad v = \dot{x} = -a\sqrt{\mu} \sin \sqrt{\mu}\, t. \tag{7}$$

Here (4) gives the velocity of the particle at any position x while (7) determines it at any time t. Relation (6) helps us to determine its position at any time.

14.2(*a*) *Discussion of the motion*

We find from (5) that the particle reaches O with the velocity $v_0 = -a\sqrt{\mu}$ [Putting $x = 0$ in (5)].

The velocity of the particle at O being non-zero, it passes through O on the other side, and immediately the acceleration alters its direction and tends to diminish the velocity. The velocity is destroyed on the left side of O at the same rate at which it was produced on the right side of O.

We find from (5) that $v = 0$ at $x = \pm a$. The particle, therefore, comes to rest at A' where $OA = OA' = a$.

It is evident from (5) that v becomes imaginary for $x > a$ and $x < -a$. Hence there is no motion outside the interval $(-a, a)$.

As the acceleration is non-zero (although $v = 0$) at $x = \pm a$, the particle stops only for a moment at A and A'. Hence, after reaching A', the particle retraces its path, passes through O and comes again to instantaneous rest at A.

The whole motion of the particle is thus an oscillation from A to A' and back, continually repeated over and over again.

Putting $x = 0$ in (6), we get $\cos \sqrt{\mu}\, t = 0$;

$$\text{or, } \sqrt{\mu}\, t = \frac{\pi}{2}; \text{ or, } \quad t = \frac{\pi}{2\sqrt{\mu}}.$$

$\therefore$ Time taken by the particle to move from A to O (or from A' to O) $= \frac{\pi}{2\sqrt{\mu}}$.

$\therefore$ Time period of a complete oscillation from A to A' and back, is given by

$$T = 4 \times \frac{\pi}{2\sqrt{\mu}} = \frac{2\pi}{\sqrt{\mu}}.$$

Time period may also be calculated by observing that (6) and (7) remain unaltered if t is replaced by $t + \frac{2\pi n}{\sqrt{\mu}}$, n being a positive integer. That is, the particle is moving through the same position with the same velocity at times $t + \frac{2\pi n}{\sqrt{\mu}}$ as at time t. Hence the motion is a periodic oscillation of period $\frac{2\pi}{\sqrt{\mu}}$.

REMARKS

1. Time period T is independent of the distance $OA = a$, i.e. independent of the point of start.

The maximum displacement of the particle on either side of O is a, and this is called the **amplitude of oscillation.** The reciprocal of time period, i.e. the number of oscillations per unit of time, namely $\frac{\sqrt{\mu}}{2\pi}$, is called the **frequency of oscillation.**

2. In Problem I, the conditions at $t = 0$, otherwise known as initial conditions, are $x = a$, $\dot{x} = 0$. In the next problem, we shall consider another set of initial conditions and see what it leads upto.

14.3 Problem II

A particle moves in a straight line OA under an attractive force towards O. The attractive force varies directly as the distance from O and the particle is initially projected with a velocity u away from O. Discuss the motion.

Let P be the position of the particle at any time t where $OP = x$, $OA = a$.

The equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -\mu x \ (x \gtrless 0, \mu, > 0). \qquad (1)$$

[The equation of motion holds on the right as well as on the left of O, as explained in Problem I above.]

The initial conditions are

$$v = \dot{x} = u \text{ when } x = a \text{ at } t = 0. \qquad (2)$$

To solve (1), we assume a trial solution

$$x = Ae^{mt},\ A \text{ and } m \text{ being constants.}$$

Substituting this value of x in (1), we have

$$Ae^{mt}(m^2 + \mu^2) = 0;\ \text{or, } m^2 = -\mu^2 = i^2\mu^2$$

$$\therefore\ m = \pm i\sqrt{\mu}$$

$$\therefore\ x = A_1 e^{i\sqrt{\mu}\,t} + B_1 e^{-i\sqrt{\mu}\,t}$$

$$= A_1(\cos\sqrt{\mu}\,t + i\sin\sqrt{\mu}\,t)$$

$$+ B_1(\cos\sqrt{\mu}\,t - i\sin\sqrt{\mu}\,t)$$

$$[\because\ e^{\mp i\theta} = \cos\theta \pm i\sin\theta]$$

$$= (A_1 - B_1)\cos\sqrt{\mu}\,t$$

$$+ i(A_1 - B_1)\sin\sqrt{\mu}\,t$$

$$= C\cos\sqrt{\mu}\,t + D\sin\sqrt{\mu}\,t, \text{ say,} \qquad (3)$$

where C and D are constants to be determined.

$$\therefore\ v = \dot{x} = -C\sqrt{\mu}\sin\sqrt{\mu}\,t +$$

$$D\sqrt{\mu}\cos\sqrt{\mu}\,t. \qquad (4)$$

By (2) and (3), we get $C = a$ while (2) and (4) give $D\sqrt{\mu} = u$; or, $D = \dfrac{u}{\sqrt{\mu}}$.

$$\therefore\ x = a\cos\sqrt{\mu}\,t + \frac{v}{\sqrt{\mu}}\sin\sqrt{\mu}\,t \qquad (5)$$

and

$$v = \dot{x} = \sqrt{\mu}\,(-a\sin\sqrt{\mu}\,t + \frac{u}{\sqrt{\mu}}\cdot$$

$$\cos\sqrt{\mu}\,t. \qquad (6)$$

Squaring (5) and (6), and then adding together, we get

$$x^2 + \frac{v^2}{\mu} = a^2 + \frac{u^2}{\mu} = (\text{say})\ b^2$$

$$\text{or,}\quad v^2 = \mu(b^2 - x^2). \qquad (7)$$

Again, writing $a = b\cos\varepsilon$, $\dfrac{u}{\sqrt{\mu}} = -b\sin\varepsilon$, we get from (5) and (6)

$$x = b\cos(\sqrt{\mu}\,t + \varepsilon) \qquad (5')$$

and

$$\frac{dx}{dt} = v = -b\sqrt{\mu}\sin(\sqrt{\mu}\,t + \varepsilon). \qquad (6')$$

Here (6) gives the velocity of the particle at any time while (7) gives it at any position. (5) gives the position of the particle at any time.

14.3(*a*) *Discussion of the motion*

The motion defined by the equation (1) and conditions (2) is:

(i) an oscillation about the centre O where $x = 0$

(ii) between the limits $x = \pm b$

(iii) of period $\dfrac{2\pi}{\sqrt{\mu}}$ which is independent of b, the maximum displacement of the particle from O on either side.

For, $\sin(\sqrt{\mu}\,t + \varepsilon)$ and $\cos(\sqrt{\mu}\,t + \varepsilon)$ are periodic functions of period 2π. If t is replaced by $t + \dfrac{2\pi n}{\sqrt{\mu}}$, we see from (5′) and (6′) that they remain unchanged when n is a positive integer. This implies that the particle is moving through the same position with the same velocity at times $t + \dfrac{2\pi}{\sqrt{\mu}}$, $t + \dfrac{4\pi}{\sqrt{\mu}}$, $t + \dfrac{6\pi}{\sqrt{\mu}}$, ...as at time t. Thus the motion is a periodic oscillation of period $\dfrac{2\pi}{\sqrt{\mu}}$.

It follows from (7) that v is imaginary outside the interval $(-b, b)$ and is zero at $x = \pm b$. As the acceleration is non-zero at $x = \pm b$, these positions are positions of instantaneous rest only. Also there is no motion outside $(-b, b)$. It is obvious that the period $\dfrac{2\pi}{\sqrt{\mu}}$ does not change if we change b by changing a and keeping μ fixed.

REMARKS

We find that the general solution of the equation of a simple harmonic motion, namely $\frac{d^2x}{dt^2} = -\mu x$ or $-n^2x$, is given by $x = b\cos(\sqrt{\mu}\,t+\epsilon)$ or $b\cos(nt+\epsilon)$ under any initial conditions; and $T = \frac{2\pi}{\sqrt{\mu}}$ or $\frac{2\pi}{n}$ gives the period of oscillation. We are now in a position to define S.H.M as follows:

Definition of Simple Harmonic motion (S.H.M.)

If a particle moves in a straight line OA in such a manner that its acceleration is always directed towards a fixed point O on it, and is at any instant proportional to its distance from O, then the motion of the particle is defined to be simple harmonic.

14.4 Illustrative examples

EXAMPLE 1

A particle is moving with simple harmonic motion and while making an excursion from one position of rest to another, its distances from the middle point of its path at three consecutive seconds are observed to be x_1, x_2 and x_3.

Prove that the time of a complete oscillation is $\frac{2\pi}{\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)}$.

Let $2a$ be the distance between the two positions of rest of the particle and let their midpoint be taken as the origin.

The distance of the particle from O at any time t is given by $x = a\cos\sqrt{\mu}\,t$ and the time of a complete oscillation is $T = \frac{2\pi}{\sqrt{\mu}}$ [see Problem I].

$$\therefore\quad x_1 = a\cos(\sqrt{\mu}\,t),$$
$$x_2 = a\cos\{\sqrt{\mu}\,(t+1)\},$$
$$x_3 = a\cos\{\sqrt{\mu}\,(t+2)\}.$$

$$\therefore\quad x_1 + x_3 = a[\cos(\sqrt{\mu}\,t) + \cos\{\sqrt{\mu}\,(t+2)\}]$$
$$= 2a\cos\{\sqrt{\mu}\,(t+1)\}\cos\sqrt{\mu}$$
$$= 2x_2\cos\sqrt{\mu}$$

or, $\cos\sqrt{\mu} = \frac{x_1+x_3}{2x_2}$;

or, $\sqrt{\mu} = \cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)$

$$\therefore\quad T = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)}$$

EXAMPLE 2

A point executes S.H.M. such that in two of its positions the velocities are u and v, and the corresponding accelerations are f_1 and f_2. Show that the distance between the positions is $\frac{v^2 \sim u^2}{f_1+f_2}$.

Prove also that the amplitude of motion is $\frac{\{(v^2-u^2)(f_1^2v^2-f_2^2u^2)\}^{\frac{1}{2}}}{f_2^2-f_1^2}$.

If 'a' be the amplitude of the S.H.M. given by $f = \frac{d^2x}{dt^2} = -\mu x$, (1)

then $V^2 = \mu(a^2 - x^2)$ (2)

where $V = \frac{dx}{dt}$ is the velocity at any time.

Let $V = u$ when $x = x_1$

and $V = v$ when $x = x_2$.

Then $f_1 = -\mu x_1$ and $f_2 = -\mu x_2$; (3)

and $u^2 = \mu(a^2 - x_1^2)$, $v^2 = \mu(a^2 - x_2^2)$. (4)

From (4), $v^2 \sim u^2 = \mu(x_1^2 \sim x_2^2)$
$$= \mu(x_1 \sim x_2)(x_1 + x_2)$$
$$= -(f_1 + f_2)(x_1 \sim x_2) \text{ by (3)}$$
$$= (f_1 + f_2)(x_2 \sim x_1).$$

$\therefore$ This distance between the two positions is $x_2 \sim x_1 = \frac{v^2 \sim u^2}{f_1+f_2}$.

Again, $v^2 - u^2 = \mu(x_1^2 - x_2^2)$ and $f_1^2 - f_2^2 = \mu^2(x_1^2 - x_2^2)$.

On dividing, $\mu = \frac{f_1^2-f_2^2}{v^2-u^2}$. (5)

Now $u^2 = \mu a^2 - \mu x_1^2 = \mu a^2 - \mu\cdot\frac{f_1^2}{u^2}$

$$= \mu a^2 - \frac{f_1^2}{\mu^2}$$

or, $u^2 + \frac{f_1^2}{\mu} = \mu a^2$

or, $a^2 = \frac{1}{\mu^2}\,[\mu u^2 + f_1^2]$. (6)

By (5), we have from (6)

$$a^2 = \left(\frac{v^2 - u^2}{f_1^2 - f_2^2}\right)^2 \left\{\frac{u^2 (f_1^2 - f_2^2) + f_1^2 (v^2 - u^2)}{v^2 - u^2}\right\}$$

$$= \frac{(v^2 - u^2)\,(v^2 f_1^2 - u^2 f_2^2)}{(f_1^2 - f_2^2)^2}$$

$\therefore$ amplitude $a = \dfrac{\{(v^2 - u^2)\,(v^2 f_1^2 - u^2 f_2^2)\}^{\frac{1}{2}}}{f_1^2 - f_2^2}$

EXAMPLE 3

A particle P moves in a straight line OCP being attracted by a force $m\mu$. PC always directed towards C, while C moves along OC with constant acceleration f. If initially C was at rest at the origin O, and P was at a distance d from O and moving with velocity V, prove that the distance of P from O at any time t is

$$\left(\frac{f}{\mu} + d\right) \cos \sqrt{\mu}\, t$$

$$+ \frac{V}{\sqrt{\mu}} \sin \sqrt{\mu}\, t - \frac{f}{\mu} + \tfrac{1}{2} f t^2$$

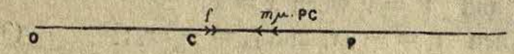


Fig. 14.3

Let $OP = x$, $OC = x_1$ at any time t.

The equation of motion of P is

$$m \frac{d^2x}{dt^2} = -m\mu (x - x_1)$$

or, $\dfrac{d^2x}{dt^2} = -\mu (x - x_1)$. (1)

Equation of motion of C is

$$\frac{d^2x_1}{dt^2} = f \quad (2)$$

where $x_1 = \frac{1}{2} f t^2$. (3)

Now (1)-(2) gives

$$\frac{d^2}{dt^2}(x - x_1) = -\mu (x - x_1) - f$$

$$= -\mu \left(x - x_1 + \frac{f}{\mu}\right)$$

or $\dfrac{d^2}{dt^2}\left(x - x_1 + \dfrac{f}{\mu}\right)$

$$= -\mu \left(x - x_1 + \frac{f}{\mu}\right).$$

The general solution of this equation is

$$x - x_1 + \frac{f}{\mu} = A \cos \sqrt{\mu}\, t + B \sin \sqrt{\mu}\, t \quad (4)$$

where A and B are constants to be determined. Differentiating (4) with respect to t,

$$\frac{dx}{dt} - \frac{dx_1}{dt} = -A\sqrt{\mu} \sin \sqrt{\mu}\, t + B\sqrt{\mu}.$$

$$.\cos \sqrt{\mu}\, t. \quad (5)$$

Now the initial conditions are

$$x = d,\ x_1 = 0,\ \frac{dx}{dt} = V,\ \frac{dx_1}{dt} = 0 \text{ at } t = 0. \quad (6)$$

By (6), we have from (4)

$$d + \frac{f}{\mu} = A.$$

By (6), we have from (5), $V = B\sqrt{\mu}$

$$\text{or } B = \frac{V}{\sqrt{\mu}}.$$

Substituting these values of A and B in (4), we get

$$x - x_1 + \frac{f}{\mu} = \left(d + \frac{f}{\mu}\right) \cos \sqrt{\mu}\, t + \frac{V}{\sqrt{\mu}}.$$

$$.\sin \sqrt{\mu}\, t$$

or, $x = \left(d + \dfrac{f}{\mu}\right) \cos \sqrt{\mu}\, t + \dfrac{V}{\sqrt{\mu}}.$

$$.\sin \sqrt{\mu}\, t - \frac{f}{\mu} + x_1$$

$$= \left(d + \frac{f}{\mu}\right) \cos \sqrt{\mu}\, t + \frac{V}{\sqrt{\mu}}.$$

$$.\sin \sqrt{\mu}\, t - \frac{f}{\mu} + \tfrac{1}{2} f t^2 \text{ by (3).}$$

This gives the distance of P from 0 at any time.

EXAMPLE 4

A particle rests in equilibrium under the attraction of two centres of forces which attract directly as the distance, their attractions per unit mass at unit distance being μ and μ'; the particle is slightly displaced towards one of them. Show that the time of a small oscillation is $\frac{2\pi}{\sqrt{\mu+\mu'}}$.

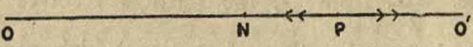


Fig. 14.4

Let the particle be in equilibrium at the position N under the attraction of two centres of forces O and O'.

Let $ON = d$ and $O'N = d'$.

Since there is equilibrium at N, $\mu d = \mu' d'$. (1)

Let the particle be slightly displaced towards (say) O' by a distance $x = NP$ in time t.

Then the equation of motion is given by

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\mu.OP + \mu'.O'P \\ &= -\mu(x+d) + \mu'(d'-x) \\ &= -\mu x - \mu' x - \mu d + \mu' d' \\ &= -(\mu + \mu')x, \text{ by (1)}. \qquad (2) \end{aligned}$$

When $x > 0$, the right hand side of (2) is negative, showing thereby that the acceleration is directed towards N.

When $x < 0$ (i.e. P is on the left of N), the right hand side of (2) is positive which shows that the acceleration is, here also, directed towards N.

Hence (2) is the equation of motion of the particle when it is displaced towards either O or O'. Equation (2) shows that the motion of the particle is simple harmonic of time period $\frac{2\pi}{\sqrt{\mu+\mu'}}$.

Exercise 14.1

1. A particle is moving in a straight line and its velocity at a distance x from the origin is $K\sqrt{a^2-x^2}$. Find the acceleration and nature of the motion.

2. At what distance from the centre will the velocity in a S.H.M. be half of the maximum velocity ?

3. V_1 and V_2 are the velocities of a particle moving in S.H.M. at distances X_1 and X_2 respectively from the centre. Show that the period of oscillation is $2\pi\{(X_1^2-X_2^2)/(V_2^2-V_1^2)\}$.

4. At the end of three successive seconds the distances of a point moving with S.H.M. from the mean position measured in the same direction are 1, 5, 5. Show that period of complete oscillation is $\frac{2\pi}{\theta}$ where $\cos\theta = \frac{3}{5}$.

5. A body performing S.H.M. in a straight line OPQ has its velocity zero when at points P and Q whose distances from O are x and y respectively, and has velocity u when halfway between them. Show that the time of a complete oscillation is $\pi(y-x)/u$.

6. The position of a particle moving in a straight line is given by $x = a\cos nt + b\sin nt$. Prove that it executes S.H.M. of period $\frac{2\pi}{n}$ and amplitude $\sqrt{a^2+b^2}$.

7. Prove that, in S.H.M., if f is the acceleration and v the velocity at any moment and T is the periodic time, then $f^2T^2 + 4\pi^2v^2$ is constant.

Find the numerical value of this constant for a motion whose periodic time is 2 secs. and in which the amplitude is 2 ft.

8. Show that a particle executing S.H.M. requires one-sixth of its period to move from the position of maximum displacement to one in which the displacement is half the amplitude.

9. A particle which is performing Simple Harmonic Motion of period T about a centre O passes through a point P with velocity V in the direction OP. Prove that the time which elapses before it returns to P is

$$\frac{T}{\pi}\tan^{-1}\left(\frac{VT}{2\pi.\,OP}\right)$$

10. A particle moving with S.H.M. has coordinates x_1 and x_2, and velocities v_1 and v_2 at any two moments. At the moment midway in time between them its coordinate and

velocity are X and V respectively. Show that

$$\frac{x_1-x_2}{v_2-v_1}=\frac{V}{\mu x},\ \frac{x_1+x_2}{v_1+v_2}=\frac{X}{V}$$

and $x_1v_2+x_2v_1=2XV$.

11. A particle oscillates with S.H.M. of amplitude a and periodic time T. Find the expression of the velocity v (i) in terms of a, T and x; (ii) in terms of a, T, t and also prove that $\int_0^T v^2 dt=\frac{2\pi^2a^2}{T}$.

12. A particle P of mass m moves in a straight line OX under the action of a force $m\mu$ (distance) directed towards a point A which moves with a constant acceleration f. Show that the motion of P is simple harmonic of period $\frac{2\pi}{\sqrt{\mu}}$ about a moving centre which is always at a distance $\frac{f}{\mu}$ behind A.

13. A particle is oscillating in a straight line about a centre of force O, towards which when at a distance x the force is mn^2x, and a is the amplitude of the oscillation; when at a distance $\frac{a\sqrt{3}}{2}$ from O, the particle receives a blow in the direction of motion which generates a velocity na. If this velocity be away from O, show that the new amplitude is $a\sqrt{3}$.

14. A particle starts from rest and moves with S.H.M. with a period of $2T$. Show that it describes $\frac{1}{4}$ of the distance before it next comes to instantaneous rest in $\frac{1}{3}$ of the time T, and attains half of its maximum velocity in $\frac{1}{6}$ of the time T.

15. Show that in S.H.M. the mean velocity (during motion from one end of the path to the other) with respect to the distance is $\frac{\pi}{4}\times$ the maximum velocity, and with respect to the time is $\frac{2}{\pi}\times$ the maximum velocity.

15 Two-Dimensional Motion

15.1 Components of velocity and acceleration parallel to two rectangular axes

Let a particle move on a curve in a plane. A point O on the plane of the curve is taken as the origin, and two mutually perpendicular lines OX, OY are taken as the axes of x and y respectively. Referred to this coordinate system, let $P(x, y)$ be the position of the particle at any time t and $Q(x + \triangle x, y + \triangle y)$ its position at time $t + \triangle t$.

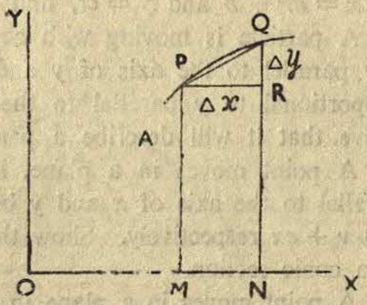


Fig. 15.1

PQ is the displacement of the particle in time $\triangle t$. Components of this displacement parallel to the x- and y-axes are $PR = \triangle x$ and $QR = \triangle y$ respectively.

Let u, v be the components of velocity of the particle parallel to OX and OY respectively.

Then u = rate of change of displacement parallel to OX

$$= \underset{\triangle t \to 0}{Lt} \frac{\triangle x}{\triangle t} = \frac{dx}{dt} \text{ or } \dot{x} \qquad (1)$$

and v = rate of change of displacement parallel to OY

$$= \underset{\triangle t \to 0}{Lt} \frac{\triangle y}{\triangle t} = \frac{dy}{dt} \text{ or } \dot{y}. \qquad (2)$$

Again, let f_x and f_y be the components of acceleration of the particle parallel to OX and OY respectively.

Then f_x = rate of change of velocity parallel to OX

$$= \frac{du}{dt} = \frac{d^2x}{dt^2} \text{ or } \ddot{x} \qquad (3)$$

and f_y = rate of change of velocity parallel to OY

$$= \frac{dv}{dt} = \frac{d^2y}{dt^2} \text{ or } \ddot{y}. \qquad (4)$$

15.2 Equations of motion

Let X, Y be the components parallel to OX, OY respectively, of the external force acting on the particle at P. If m be the mass of the particle, its equations of motion parallel to OX and OY respectively are

$$m\frac{d^2x}{dt^2} = X, \qquad \text{(i)}$$

$$\text{and} \quad m\frac{d^2y}{dt^2} = Y. \qquad \text{(ii)}$$

15.2(*a*) *Illustrative examples*

EXAMPLE 1

The position of a moving particle at time t is given by $x = a\cos\mu t$, $y = a\sin\mu t$. Find the velocity, acceleration and path of the particle.

The components of velocity of the particle parallel to the x- and y-axes are respectively

$u = \dot{x} = -a\mu \sin \mu t$ and $v = \dot{y} = a\mu \cos \mu t$.

$\therefore$ Velocity of the particle at any time is

$$w = \sqrt{u^2 + v^2} = \sqrt{a^2\mu^2 (\sin^2 \mu t + \cos^2 \mu t)}$$
$$= a\mu.$$

Components of acceleration of the particle parallel to the x- and y-axes are respectively

$$f_x = \ddot{x} = -a\mu^2 \cos \mu t$$

and $f_y = \ddot{y} = -a\mu^2 \sin \mu t$.

$\therefore$ Acceleration of the particle at any time is

$$f = \sqrt{f_x^2 + f_y^2} = a\mu^2.$$

Hence the particle moves with constant velocity and constant acceleration.

Eliminating time t from $x = a \cos \mu t$ and $y = a \sin \mu t$, we get $x^2 + y^2 = a^2$ which is the equation to the path of the particle.

EXAMPLE 2

A particle moves on a plane under a force having components X, Y per unit of mass parallel to rectangular axes (X, Y being functions of x and y), show that the differential equation to the path is

$$\frac{d}{dx}\left[\left(Y - X\frac{dy}{dx}\right) \Big/ \frac{d^2y}{dx^2}\right] = 2X.$$

The equations of motion of the particle are $\ddot{x} = X$ and $\ddot{y} = Y$. (1)

$$\text{Now} \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}}. \qquad (2)$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$= \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)\frac{dt}{dx}$$
$$= \frac{\frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)}{\frac{dx}{dt}} = \frac{\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2}}{\dot{x}}$$
$$= \frac{\dot{x}\left(\ddot{y} - \frac{\dot{y}}{\dot{x}}\ddot{x}\right)}{\dot{x}^3}$$

$$\text{or,} \quad \frac{d^2y}{dx^2} = \frac{\dot{x}\left(Y - X\frac{dy}{dx}\right)}{\dot{x}^3} \quad \text{by (1) and (2)}$$
$$= \frac{Y - X\frac{dy}{dx}}{\dot{x}^2}.$$

$$\therefore \ \dot{x}^2 = \frac{Y - X\frac{dy}{dx}}{\frac{d^2y}{dx^2}}$$

$$\therefore \frac{d}{dx}\left[\left(Y - X\frac{dy}{dx}\right) \Big/ \frac{d^2y}{dx^2}\right]$$
$$= \frac{d}{dx}\left(\dot{x}^2\right) = \frac{d}{dt}\left(\dot{x}^2\right)\frac{dt}{dx} = 2\dot{x}\ddot{x}.\frac{1}{\dot{x}}$$
$$= 2\ddot{x} = 2X.$$

Hence the result.

Exercise 15.1

1. If the position of a moving point is given by $x = at^2 + b$ and $y = ct$, find the path.
2. A particle is moving with constant velocity parallel to the axis of y and a velocity proportional to y parallel to the axis of x. Prove that it will describe a parabola.
3. A point moves in a plane, its velocities parallel to the axis of x and y being $u + ey$ and $v + ex$ respectively. Show that it moves in a conic section.
4. A point moves in a plane in such a way that its co-ordinates x, y are given at any time t by $x = a \cos \theta t$, $y = b \sin \theta t$ (θ = constant).

   Show that the path is an ellipse, and find the acceleration in terms of the distance from the origin; also determine the direction of the acceleration at any time t.
5. If the coordinates (x, y) of a moving point are given by $x = a(\cos\theta + \theta \sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$, and θ increases at a uniform rate w, prove that the velocity of the point is $a\theta w$; and find the inclination of the velocity to the axis of x.
6. The coordinates of a moving point at time t are given by $x = a(2t + \sin 2t)$, $y = a(1 - \cos 2t)$. Prove that its acceleration is constant and find the direction of motion at time t.
7. A particle moves freely in a semi-circle

under a force directed always perpendicularly towards the bounding diameter. Show that the force varies inversely as the cube of the ordinate to the diameter.

8. A particle describes a parabola under a force which is always directed perpendicularly towards its axis. Prove that the force must be inversely proportional to the cube of the ordinate.

15.3 Motion of a projectile in vacuo

We shall consider the motion of a particle which is projected in a vertical plane in any direction, assuming that air offers no resistance. Here the particle is called a **projectile** and the curve described by it is called the **path of the projectile** or its **trajectory.**

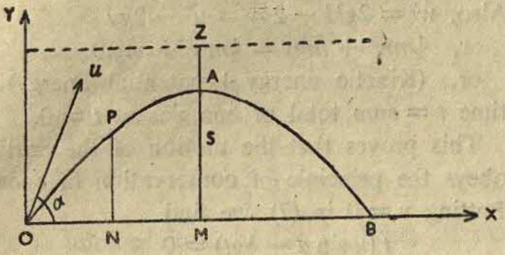


Fig. 15.2

Let a particle of mass m be projected in a vertical plane from O with an initial velocity u at an angle α to the horizon. We take the point O as the origin, and the horizontal and the vertical directions through O as the axes of x and y respectively.

Let $P(x, y)$ be the position of the particle at any time t.

Equations of motion of the particle in the horizontal and the vertical directions are respectively

$$m\ddot{x} = 0 \text{ and } m\ddot{y} = -mg$$

or, $\ddot{x} = 0$ (1)

and $\ddot{y} = -g.$ (2)

The initial conditions are

$x = y = 0, \dot{x} = u\cos\alpha, \dot{y} = u\sin\alpha$ at $t = 0$. (3)

Integrating (1), we get $\dot{x} = C$.

By (3), $C = u\cos\alpha$.

$\therefore\ \dot{x} = u\cos\alpha.$ (4)

Relation (4) shows that the horizontal component of velocity remains constant throughout the motion of the particle.

Integrating (2) $\dot{y} = C' - gt$.

By (3), $u\sin\alpha = C'$.

$\therefore\ \dot{y} = u\sin\alpha - gt.$ (5)

(5) gives the vertical component of velocity at any time.

Integrating (4) and (5) again, we get

$$x = k + ut\cos\alpha \text{ and } y = k_1 + ut\sin\alpha - \tfrac{1}{2}gt^2.$$

Using (3), we find $k = 0$, $k_1 = 0$.

$\therefore\ x = ut\cos\alpha$ (6)

and $y = ut\sin\alpha - \frac{1}{2}gt^2.$ (7)

Relations (6) and (7) together define the position of the particle at any time t.

From (6), $t = \dfrac{x}{u\cos\alpha}$.

Substituting this value of t in (7), we have

$$y = x\tan\alpha - \tfrac{1}{2}\cdot\frac{gx^2}{u^2\cos^2\alpha} \tag{8}$$

This gives the equation to the path of the particle. Equation (8) may be written as

$$\frac{2u^2\cos^2\alpha}{g}y = \frac{x\sin\alpha}{\cos\alpha}\times\frac{2u^2\cos^2\alpha}{g} - x^2$$

$$\text{or, } x^2 - \frac{u^2\sin 2\alpha}{g}x = -\frac{2u^2\cos^2\alpha}{g}y$$

$$\text{or, } x^2 - \frac{u^2\sin 2\alpha}{g}x + \left(\frac{u^2}{2g}\sin 2\alpha\right)^2 = -\frac{2u^2\cos^2\alpha}{g}y + \frac{u^4\sin^2 2\alpha}{4g^2}$$

$$\text{or, } \left(x - \frac{u^2}{2g}\sin 2\alpha\right)^2 = -\frac{2u^2\cos^2\alpha}{g}\cdot\left(y - \frac{u^2\sin^2\alpha}{2g}\right)$$

Transferring the origin to the point

$$\left(\frac{u^2}{2g}\sin 2\alpha,\ \frac{u^2\sin^2\alpha}{2g}\right),$$

the above equation reduces to the form

$$X^2 = -\frac{2u^2\cos^2\alpha}{g}Y.$$

This shows that the trajectory is a parabola with its axis along the negative y-axis, i.e. vertically downwards. The length of the latus rectum is $\frac{2u^2 \cos^2 \alpha}{g}$ and the coordinates of the vertex A are

$$\left(\frac{u^2}{2g} \sin 2\alpha, \frac{u^2}{2g} \sin^2 \alpha \right).$$

The axis being vertical, the directrix of the parabolic trajectory is horizontal.

The maximum height attained by the particle is

$$AM = \text{ordinate of the point } A = \frac{u^2}{2g} \sin^2 \alpha.$$

If the particle reaches the horizon again at B, OB is called the **range** R **of the projectile.**

$$\therefore \quad R = OB = 2OM$$
$$= 2 \times \text{abscissa of the point } A$$
$$= 2 \times \frac{u^2}{2g} \sin 2\alpha = \frac{u^2}{g} \sin 2\alpha.$$

The range will be maximum when $\sin 2\alpha = 1$ or $\alpha = \frac{\pi}{4}$ and then $R_{\max} = \frac{u^2}{g}$.

Since R remains unaltered when α is replaced by $\left(\frac{\pi}{2} - \alpha \right)$, it is evident that for the same range there can be two angles of projection namely α and $\frac{\pi}{2} - \alpha$.

We know from Coordinate Geometry,

$$AS = AZ = \tfrac{1}{4} \times \text{latus rectum}$$
$$= \frac{u^2 \cos^2 \alpha}{2g}$$

where S is the focus of the trajectory, and Z is the point of intersection of its axis and directrix.

$\therefore$ The height of the directrix above the point of projection is

$$H = MZ = AM + AZ$$
$$= \frac{u^2}{2g} \sin^2 \alpha + \frac{u^2}{2g} \cos^2 \alpha$$
$$= \frac{u^2}{2g}.$$

The height of the focus S above OX is

$$SM = AM - AS = \frac{u^2}{2g} \sin^2 \alpha - \frac{u^2}{2g} \cos^2 \alpha$$
$$= -\frac{u^2}{2g} \cos 2\alpha.$$

$\therefore$ coordinates of the focus S are

$$\left(\frac{u^2}{2g} \sin 2\alpha, -\frac{u^2}{2g} \cos 2\alpha \right).$$

If w be the velocity of the projectile at P, then

$$w^2 = \dot{x}^2 + \dot{y}^2$$
$$= u^2 \cos^2 \alpha + (u \sin \alpha - gt)^2$$
$$= u^2 - 2ugt \sin \alpha + g^2 t^2$$
$$= 2g \left[\frac{u^2}{2g} - (ut \sin \alpha - \tfrac{1}{2} g t^2) \right]$$
$$= 2g \, [H - y]$$
$$= 2g \times \text{depth of the particle below the directrix.}$$

Also, $w^2 = 2gH - 2gy = u^2 - 2gy$

or, $\frac{1}{2} m w^2 + mgy = \frac{1}{2} m u^2 + mg.0$

or, (Kinetic energy + potential energy) at time t = sum total of energies at $t = 0$.

This proves that the motion of the particle obeys the principle of conservation of energy.

Putting $y = 0$ in (7), we find

$$t \, (u \sin \alpha - \tfrac{1}{2} gt) = 0$$

so that $t = 0$ or $t = \frac{2u \sin \alpha}{g}$.

Here $t = 0$ gives the time of projection of the particle, and $t = \frac{2u \sin \alpha}{g}$ is the time to come back (at B) to the plane of projection.

The time $T = \frac{2u \sin \alpha}{g}$ is known as the **time of flight.**

15.4 Illustrative examples

EXAMPLE 1

A heavy particle is projected from a point O at an angle α to the horizon and describes a parabola under gravity. If coordinate axes are taken horizontally and vertically through O, prove that the equation of the parabola is

$$y = x\tan\alpha\left(1 - \frac{x}{R}\right)$$

where R is the horizontal range.

If the distance between the two points on the parabola which are at the same height h above the horizontal is $2a$, show that

(i) $R(R - 4h\cot\alpha) = 4a^2$.

(ii) $R = 2a\cot\frac{\alpha}{2}$ when $h = a$.

We know that the equation to the parabolic path is $y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$ (1)

and the range $R = \dfrac{u^2}{g}\sin 2\alpha$. (2)

From (1),

$$\therefore\quad y = x\tan\alpha\left(1 - \frac{gx^2}{2u^2\cos\alpha\sin\alpha}\right)$$

$$\text{or,}\quad y = x\tan\alpha\left(1 - \frac{gx}{u^2\sin 2\alpha}\right)$$

$$= x\tan\alpha\left(1 - \frac{x}{\frac{u^2}{g}\sin 2\alpha}\right)$$

$$= x\tan\alpha\left(1 - \frac{x}{R}\right), \text{ by (2).} \tag{3}$$

This is another form of the equation of the parabola (1).

(i) Let x be the abscissa of the point corresponding to a height h. Then (x, h) satisfies equation (1).

$$\therefore\quad h = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

or, $gx^2 - 2u^2\cos\alpha\sin\alpha . x + 2hu^2\cos^2\alpha = 0$.

This being a quadratic equation in x, it has two roots x_1, x_2. This means $Q(x_1, h)$, $Q_1(x_2, h)$ are two points on the parabola (1) at the same height h.

$$\therefore\quad x_1 + x_2 = \frac{2u^2\cos\alpha\sin\alpha}{g} = \frac{u^2}{g}\sin 2\alpha = R \tag{4}$$

$$\text{and } x_1x_2 = \frac{2hu^2\cos^2\alpha}{g}. \tag{5}$$

By the given condition, $x_2 - x_1 = 2a$.

$$\therefore\quad (x_2 + x_1)^2 - 4x_1x_2 = 4a^2$$

$$\text{or,}\quad R^2 - \frac{8hu^2\cos^2\alpha}{g} = 4a^2. \tag{6}$$

From (4), $\dfrac{2u^2\cos\alpha}{g} = \dfrac{R}{\sin\alpha}$.

Using this result, we get from (6),

$$R^2 - 4h\cos\alpha . \frac{R}{\sin\alpha} = 4a^2$$

$$\text{or,}\quad R(R - 4h\cot\alpha) = 4a^2. \tag{7}$$

(ii) Putting $h = a$ in (7), we get

$$R^2\sin\alpha - 4aR\cos\alpha - 4a^2\sin\alpha = 0$$

$$\therefore\quad R = \frac{4a\cos\alpha \pm \sqrt{16a^2\cos^2\alpha + 16a^2\sin^2\alpha}}{2\sin\alpha}$$

$$\text{or,}\quad R = \frac{4a(\cos\alpha + 1)}{2\sin\alpha} = \frac{2a . \cancel{2}\cos^2\frac{\alpha}{2}}{\cancel{2}\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

$$= 2a\cot\frac{\alpha}{2}$$

Since $\cos\alpha < 1$, the negative sign is rejected, for otherwise R will be negative.

EXAMPLE 2

A particle is projected so as to have a range R on a horizontal plane through the point of projection and the greatest height attained by it is H. Prove that the initial velocity of projection is

$$\left\{2g\left(H + \frac{R^2}{16H}\right)\right\}^{\frac{1}{2}}$$

and that the maximum horizontal range is

$$2H + \frac{R^2}{8H}.$$

Let u be the velocity of projection and α the angle of projection.

The horizontal range is

$$R = \frac{2u^2\sin\alpha\cos\alpha}{g} = \frac{u^2}{g}\sin 2\alpha \tag{1}$$

The greatest height is

$$H = \frac{u^2\sin^2\alpha}{2g} \tag{2}$$

$$\text{Now}\quad 2H + \frac{R^2}{8H} = \frac{u^2\sin^2\alpha}{g} + \frac{\cancel{4}u^4\sin^2\alpha\cos^2\alpha}{g^2}$$

$$\times \frac{2g}{\underset{\cancel{4}}{\cancel{8}}\, u^2 \sin^2 \alpha}$$

$$= \frac{u^2 \sin^2 \alpha}{g} + \frac{u^2 \cos^2 \alpha}{g}$$

$$= \frac{u^2}{g}.$$

∴ The maximum horizontal range is

$$R_{\max} = \frac{u^2}{g} \text{ (when } \sin 2\alpha = 1)$$

$$= 2H + \frac{R^2}{8H}. \qquad (3)$$

From (ii), $\operatorname{cosec}^2 \alpha = \frac{u^2}{2gH}$.

Dividing (1) by (2), $\frac{R}{H} = 4 \cot\alpha$;

$$\text{or, } \cot\alpha = \frac{R}{4H}.$$

$$\therefore \operatorname{cosec}^2 \alpha = 1 + \cot^2 \alpha = 1 + \frac{R^2}{16H^2}$$

$$\therefore \frac{u^2}{2gH} = 1 + \frac{R^2}{16H^2}$$

$$\text{or, } u^2 = 2g\,H + \left(\frac{R^2}{16H}\right).$$

$$\therefore u = \left\{2g\left(H + \frac{R^2}{16H}\right)\right\}^{\frac{1}{2}}$$

Hence the result.

EXAMPLE 3

An aeroplane flying with a constant velocity u, at a constant height h, passes directly over a gun. When the elevation of the aeroplane above the horizontal plane is θ as seen from the gun, the gun is fired point blank at it. Show that the shot hits the aeroplane if $2(V \cos \theta - u)\, u \tan^2 \theta = gh$, where V is the initial velocity of the shot, its path being parabolic.

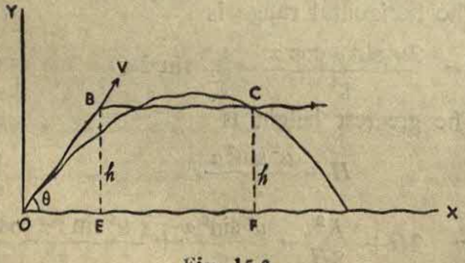


Fig. 15.3

Let the gun be fired from O when the aeroplane is at B, and let C be the position of the aeroplane when the shot hits it.

The horizontal and the vertical directions through O are taken as the axes of x and y respectively. Then the position of the shot at any time t is given by

$$x = Vt \cos \theta \qquad (1)$$

$$\text{and} \qquad y = Vt \sin \theta - \tfrac{1}{2} g t^2. \qquad (2)$$

Let BE and CF be drawn perpendicular to the axis of x. Then $BE = CF = h$ and $\angle BOE = \theta$.

$$\therefore OE = BE \cot \theta = h \cot \theta.$$

If the shot hits the aeroplane in time T after it is fired, we get

$$OF = VT \cos \theta \text{ by (1),} \qquad (3)$$

$$\text{and } h = VT \sin \theta - \tfrac{1}{2} g T^2 \text{ by (2).} \qquad (4)$$

From (3), $VT \cos \theta = OF = OE + EF$

$$= h \cot \theta + uT$$

$$\text{or, } (V \cos \theta - u)\, T = h \cot \theta$$

$$\therefore T = \frac{h \cot \theta}{(V \cos \theta - u)}.$$

Substituting this value of T in (4), we get

$$h = V \sin \theta . \frac{h \cos \theta}{\sin \theta\, (V \cos \theta - u)} - \frac{g h^2 \cot^2 \theta}{2\, (V \cos \theta - u)^2}$$

$$\text{or, } \frac{g h^2 \cot^2 \theta}{2 (V \cos \theta - u)^2} = h \left[\frac{V \cos \theta}{V \cos \theta - u} - 1\right] = \frac{hu}{V \cos \theta - u}$$

$$\text{or, } \frac{gh}{2\, (V \cos \theta - u) \tan^2 \theta} = u$$

$$\therefore 2\, (V \cos \theta - u)\, u \tan^2 \theta = gh.$$

EXAMPLE 4

Two shots are projected from a gun at the top of a hill with the same velocity u at the angles of projection α and β respectively. If the shots strike the horizontal ground through the foot of the hill at the same point, show that the height h of the hill above the plane is given by

$$h = \frac{2u^2(1 - \tan\alpha\tan\beta)}{g(\tan\alpha + \tan\beta)^2}.$$

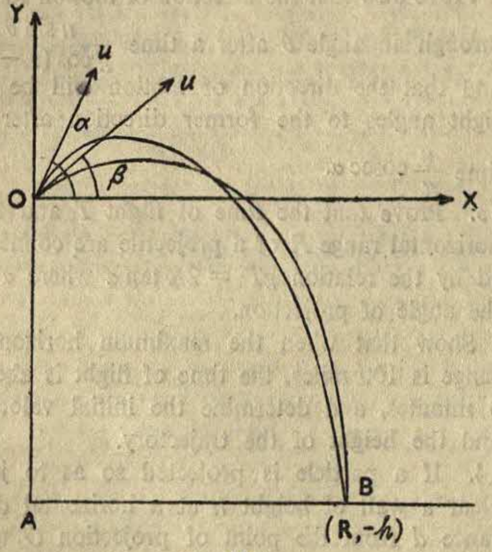


Fig. 15.4

Let the shots be fired from a gun stationed at the top O of a hill AO of height h. O is taken as the origin, and the horizontal and vertical directions through O are taken as the axes of x and y respectively. If R be the horizontal range for both the shots, the point struck by the shots is $B(R, -h)$ on the horizontal ground through the foot A of the hill.

We know that the equation to the path of the shots is

$$y = x\tan\theta - \tfrac{1}{2}\cdot\frac{gx^2}{u^2\cos^2\theta} \qquad (1)$$

where θ is the angle of projection.

Since the point B lies on the path of each shot, we have

$$-h = R\tan\alpha - \frac{gR^2}{2u^2\cos^2\alpha} \qquad (2)$$

and $$-h = R\tan\beta - \frac{gR^2}{2u^2\cos^2\beta}. \qquad (3)$$

Subtracting (3) from (2),

$$R(\tan\alpha - \tan\beta) = \frac{gR^2}{2u^2}(\sec^2\alpha - \sec^2\beta)$$

or, $$\frac{2u^2}{g}(\tan\alpha - \tan\beta) = R\{(1 + \tan^2\alpha) - (1 + \tan^2\beta)\}$$

or, $$\frac{2u^2}{g} = R(\tan\alpha + \tan\beta)$$

$$\therefore\ R = \frac{2u^2}{g(\tan\alpha + \tan\beta)}. \qquad (4)$$

Putting the value of R in (2), we get

$$h = R\left[\frac{gR}{2u^2\cos^2\alpha} - \tan\alpha\right]$$

$$-\frac{2u^2}{g(\tan\alpha + \tan\beta)}\left[\frac{\sec^2\alpha}{\tan\alpha + \tan\beta} - \tan\alpha\right]$$

$$-\frac{2u^2}{g(\tan\alpha + \tan\beta)^2}\cdot[\sec^2\alpha - \tan^2\alpha - \tan\alpha\tan\beta]$$

$$-\frac{2u^2}{g(\tan\alpha + \tan\beta)^2}\left[1 - \tan\alpha\tan\beta\right]$$

Hence the result.

Exercise 15.2

1. A particle is projected with a velocity of 960 ft./sec. at an elevation 30°. Find (i) the greatest height attained, (ii) the time of flight and the range on a horizontal plane through the point of projection, (iii) the velocity and direction of motion at a height of 528 ft.
2. A projectile thrown from a point in a horizontal plane comes back to the plane in 4 seconds at a distance of 64 yds. from the point of projection. Find the velocity of projection in feet per sec.
3. A particle is projected with a velocity of 48 ft./sec. Find the maximum range on a horizontal plane through the point of projection and the two directions of projection to give a range of 36 ft.
4. A shot is fired from a gun on the top of a vertical cliff 400 ft. high, with a velocity of 768 ft. per sec. at an elevation of 30°. Find the horizontal distance from the foot of

the cliff of the point where the shot strikes the water.

5. A tennis ball served horizontally from a height 6.25 ft. strikes the ground at a point 60 ft. away from the server. If it just strikes the net 40 ft. away from the server, find the height of the net.

6. Show that a particle starting with a velocity of 100 ft./sec. at an angle $\tan^{-1}\frac{3}{4}$ to the horizon will just clear a wall 36 ft. high at a horizontal distance of 80 yds. from the point of projection.

7. A cricket ball thrown from a height of 6 ft. at an angle of 30° with the horizon, with a speed of 60 ft./sec., is caught by another fieldsman, at a height of 2 ft. from the ground. How far apart were the two men ?

8. A particle is projected so as to pass through two points whose horizontal distances from the point of projection are 36 and 72 ft., and which are at vertical heights 11 and 14 ft. above the horizontal plane through the point of projection. Find the velocity and direction of projection.

9. A bombshell on striking the ground (supposed to be horizontal) bursts, scattering its fragments with velocities of magnitude u in different directions ; find the area of the ground covered by the fragments.

10. A projectile started from O at an elevation α. After t seconds its position appeared to have an elevation β as seen from O. Prove that the initial velocity of the projectile was $\dfrac{gt\cos\beta}{2\sin(\alpha-\beta)}$.

11. A particle is projected in a direction making an angle θ with the horizon. If it passes through the point (x_1, y_1), show that $\tan\theta = \dfrac{y_1}{x_1}\cdot\dfrac{R}{R-x_1}$ and that if it passes through the point (x_2, y_2) also, then $\tan\theta = \dfrac{x_2^2 y_1 - x_1^2 y^2}{x_1 x_2 (x_2 - x_1)}$ where R denotes the horizontal range.

12. A particle is projected with a velocity u at an elevation α. After a time t, the direction of its path makes an angle β with the horizon. Prove that $u\cos\alpha = \dfrac{gt}{\tan\alpha - \tan\beta}$.

Prove also that the direction of motion turns through an angle θ after a time $\dfrac{u\sin\theta}{g\cos(\theta-\alpha)}$ and that the direction of motion will be at right angles to the former direction after a time $\dfrac{u}{g}\operatorname{cosec}\alpha$.

13. Prove that the time of flight T and the horizontal range R of a projectile are connected by the relation $gT^2 = 2R\tan\alpha$ where α is the angle of projection.

Show that when the maximum horizontal range is 100 miles, the time of flight is about 3 minutes, and determine the initial velocity and the height of the trajectory.

14. If a particle is projected so as to just clear a wall of height h at a horizontal distance d from the point of projection O and to have a range R on the horizontal plane through O, show that the velocity of projection u is given by $\dfrac{2u^2}{g} = \dfrac{d^2(R-d)^2 + h^2R^2}{dh(R-d)}$.

15. A particle is projected with an initial velocity u. If the greatest height attained by the particle be H, prove that the range on the horizontal plane through the point of projection is $R = 4\left\{H\left(\dfrac{u^2}{2g} - H\right)\right\}^{\frac{1}{2}}$.

16. Shots fired simultaneously from the top and bottom of a vertical cliff with elevations α and β respectively strike an object simultaneously at the same point. Show that if a is the horizontal distance of the object from the cliff, its height is $a(\tan\beta - \tan\alpha)$.

17. Two bodies are projected from the same point O in directions making angles α_1 and α_2 with the horizon and they pass through the point A in the horizontal plane through O. Prove that if t_1 and t_2 be their times of flight

from O to A, then $\dfrac{t_1^2 - t_2^2}{t_1^2 + t_2^2} = \dfrac{\sin(\alpha_1 - \alpha_2)}{\sin(\alpha_1 + \alpha_2)}$.

18. A ball is projected from a point on the ground distant a from the foot of the vertical wall of height b, the velocity of projection being u at an angle α to the horizon. Find how high above the wall the ball passes it. If the ball just clears the wall, prove that the greatest height reached is $\dfrac{a^2 \tan^2 \alpha}{4(a \tan \alpha - b)}$.

19. A ball is projected so as to clear two walls, the first of height a at a distance b from the point of projection and the second of height b at a distance a from the point of projection. Show that the range on the horizontal plane is $\dfrac{a^2 + ab + b^2}{a + b}$ and that the angle of projection exceeds $\tan^{-1} 3$.

20. It t_1 and t_2 are the two times of flight with which a given range R on a horizontal plane can be reached by a particle projected with velocity u, prove that t_1, t_2 are the roots of the equation $g^2t^4 - 4u^2t^2 + 4R^2 = 0$.

Deduce from above that if the fragments of a shell fly at all angles with a speed of $5g$ ft./sec., a man at a distance of $24g$ ft. from the point of explosion is in danger for 2 secs.

STATICS

16 Concurrent Forces

16.1 Resultant of two concurrent forces

The resultant R of two given forces P and Q acting at a point A is given by the law of **Parallelogram of Forces** which is stated below :

> If two forces P and Q acting at a point A be represented in magnitude, direction and sense by the sides of a parallelogram drawn through A, then their resultant R is represented in magnitude, direction and sense by the diagonal of the parallelogram through A [See Fig. 16.1].

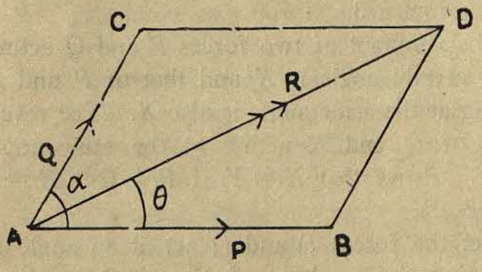


Fig. 16.1

16.2 To find the resultant of two given forces acting at a point

Let two forces P and Q act at an angle α (acute or obtuse) at the point A. The magnitude of resultant R of P and Q is given by the formula

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha} \qquad (1)$$

The direction of R is given by

$$\theta = \tan^{-1} \frac{Q \sin \alpha}{P + Q \cos \alpha} \qquad (2)$$

where θ is the angle made by the direction of R with that of P [See Fig. 16.1].

REMARK

The proof is similar to that in section 11.4 of Ch. 11, where u, v, w are to be replaced by P, Q, R respectively.

COROLLARY

1. We have from (1), $R = P + Q$ when $\alpha = 0$ and $R = P \sim Q$ when $\alpha = \pi$.

Hence *the resultant of two given forces acting along the same line is their algebraic sum.*

2. The greatest resultant of P and Q is $P + Q$ when $\alpha = 0$ (i.e. when P, Q act along the same line in the same direction) and the least resultant is $P \sim Q$ when $\alpha = \pi$ (i.e. when P and Q act along the same line in opposite directions).

3. When $P = Q$, we have $R = 2P \cos \frac{1}{2}\alpha$ from (1) and $\theta = \frac{1}{2}\alpha$ from (2). Hence the resultant of two equal forces P, P acting at an angle α is $2P \cos\frac{\alpha}{2}$ and it bisects the angle between them.

16.3 Illustrative examples

EXAMPLE 1

Two forces of magnitudes $3P$, $2P$ respectively have a resultant R. If the first force is doubled, the magnitude of the resultant is doubled. Find the angle between the forces.

Let the forces $3P$ and $2P$ act at an angle α. Then

$$R^2 = (3P)^2 + (2P)^2 + 2 \,.\, (3P) \,.\, (2P) \cos\alpha$$
$$= 9P^2 + 4P^2 + 12P^2 \cos\alpha$$
$$= 13P^2 + 12P^2 \cos\alpha$$
$$= P^2(13 + 12 \cos\alpha) \qquad (1)$$

Let R_1 be the resultant of forces $6P$, $2P$ acting at the same angle α.

Then

$$R_1^2 = (6P)^2 + (2P)^2 + 2(6P)(2P) \cos\alpha$$
$$= 36P^2 + 4P^2 + 24P^2 \cos\alpha$$
$$= P^2(40 + 24 \cos\alpha) \qquad (2)$$

By the given condition, $R_1 = 2R$

$$\therefore \quad R_1^2 = 4R^2$$

or, $P^2(40 + 24\cos\alpha) = 4P^2(13 + 12 \cos\alpha)$

or, $40 + 24 \cos\alpha = 52 + 48 \cos\alpha$

or, $24 \cos\alpha - 48 \cos\alpha = 52 - 40$

or, $-24 \cos\alpha = 12$

or, $\cos\alpha = -\frac{1}{2} = \cos 120°$

$$\therefore \quad \alpha = 120°$$

Hence the angle between the forces $3P$, $2P$ is $120°$.

EXAMPLE 2

Two forces $P + Q$, $P - Q$ make an angle 2α with one another, and their resultant makes an angle θ with the bisector of the angle between them. Show that

$$P \tan\theta = Q \tan\alpha.$$

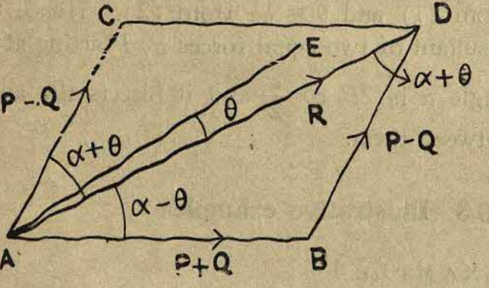


Fig. 16.2

Let the forces $P + Q$ and $P - Q$ be represented, in magnitude and direction, by the sides AB and AC respectively of the parallelogram $ABDC$ so that their resultant R is represented by AD.

AE bisects $\angle BAC$. $\quad \therefore \ \angle DAE = \theta$ and $\angle BAE = \alpha$.

Hence $\angle BAD = \alpha - \theta$ and $\angle BDA = \alpha + \theta$ as shown in Fig. 16.2.

By applying sine formula to ΔABD,

$$\frac{BD}{\sin BAD} = \frac{AB}{\sin BDA}$$

or, $$\frac{P - Q}{\sin(\alpha - \theta)} = \frac{P + Q}{\sin(\alpha + \theta)}$$

or, $$\frac{P + Q}{P - Q} = \frac{\sin(\alpha + \theta)}{\sin(\alpha - \theta)}$$

By componendo and dividendo, we get

$$\frac{2P}{2Q} = \frac{\sin(\alpha + \theta) + \sin(\alpha - \theta)}{\sin(\alpha + \theta) - \sin(\alpha - \theta)}$$

or, $$\frac{P}{Q} = \frac{2 \sin\alpha \cos\theta}{2 \cos\alpha \sin\theta} = \frac{\tan\alpha}{\tan\theta}$$

$$\therefore \quad P \tan\theta = Q \tan\alpha.$$

EXAMPLE 3

The resultant of two forces P and Q acting at a certain angle is X and that of P and R acting at the same angle is also X. The resultant of Q and R acting at the same angle is Y. Prove that $X = Y$ if $P + Q + R = 0$ $(Q \neq R)$.

Let the forces P and Q act at an angle α. X being the resultant of P and Q, we have

$$X^2 = P^2 + Q^2 + 2PQ \cos\alpha$$

or, $X^2 - P^2 - Q^2 = 2PQ \cos\alpha \qquad (1)$

Similarly,

$$X^2 - P^2 - R^2 = 2PR \cos\alpha \qquad (2)$$

and $Y^2 - Q^2 - R^2 = 2QR \cos\alpha. \qquad (3)$

Subtracting (2) from (1),

$$R^2 - Q^2 = 2P(Q - R) \cos\alpha$$

or, $-(Q + R)(Q - R) = 2P(Q - R) \cos\alpha$

or, $-(Q + R) = 2P \cos\alpha \quad [\because \ Q \neq R]$

$$\therefore \quad \cos\alpha = -\frac{Q + R}{2P} \qquad (4)$$

If $P + Q + R = 0$, $Q + R = -P$ and hence from (4),

$$\cos \alpha = -\frac{-P}{2P} = \frac{1}{2}.$$

Substituting the value of $\cos \alpha$ in (2) and (3),

$$X^2 = P^2 + R^2 + PR$$

and $Y^2 = Q^2 + R^2 + QR$

Subtracting these relations,

$$X^2 - Y^2 = P^2 - Q^2 + R(P - Q)$$

or, $(X + Y)(X - Y) = (P - Q)(P + Q + R)$

or, $X - Y = \dfrac{(P - Q)(P + Q + R)}{X + Y}$

$= 0$ since $P + Q + R = 0$

$\therefore \quad X = Y$

Exercise 16.1

1. Two forces whose magnitudes are P and $P\sqrt{2}$ lbs. wt. act on a particle in directions inclined at an angle of 135° to each other; find the magnitude and direction of the resultant.
2. Find the angle between two equal forces P, when their resultant is a third equal force P.
3. Find the greatest and least resultants of two forces whose magnitudes are 12 and 8 lbs. wt. respectively.
4. Two equal forces act on a particle; find the angle between them when the square of their resultant is equal to three times their product.
5. Show that the greater the angle between the lines of action of two forces acting at a point, the less will be their resultant.
6. If the direction and magnitude of the resultant of two forces inclined at a given obtuse angle be known, prove that one of the forces is greatest when the other is perpendicular to the resultant.
7. Two unequal forces inclined at a certain angle act on a particle. Show that the resultant is nearer the greater force.
8. If the resultant of two forces acting on a particle be at right angles to one of them, and its magnitude be one-third of the magnitude of the other, show that the ratio of the larger force to the smaller is $3 : 2\sqrt{2}$.
9. The resultant of two forces P and Q acting at an angle α is equal to $(2m + 1)\sqrt{P^2 + Q^2}$; when the forces act at an angle $(90° - \alpha)$, the resultant is $(2m - 1)\sqrt{P^2 + Q^2}$. Show that $\tan \theta = \dfrac{m - 1}{m + 1}$.
10. If the resultant R of two forces P and Q inclined to one another at any given angle make an angle α with the direction of P, show that the resultant of the forces $P + R$ and Q acting at the same angle will make an angle $\frac{1}{2}\alpha$ with the direction of $P + R$.
11. Two forces P and Q act at an angle α. If P and Q be interchanged in position, show that the resultant will be turned through an angle θ where $\tan \dfrac{\theta}{2} = \dfrac{P - Q}{P + Q} \tan \dfrac{\alpha}{2}$.
12. Two forces P and Q act at an angle α and have a resultant R. If the magnitude of each force be increased by R, and ϕ be the angle which the new resultant makes with R, prove that
$$\tan \phi = \frac{(P - Q)\sin \alpha}{P + Q + R + (P + Q)\cos \alpha}.$$
13. The resultant of two forces P and Q is R. If Q is doubled, R is doubled and when Q is reversed, R is again doubled. Show that $P : Q : R : : \sqrt{2} : \sqrt{3} : \sqrt{2}$.
14. The line of action of the resultant of two forces P and Q divides the angle between them in the ratio 1 : 2. Show that the magnitude of their resultant is $(P^2 - Q^2)/Q$.
15. The greatest and least magnitudes of the resultant of two forces of constant magnitudes are F and G. Prove that when the forces act at an angle 2α, the resultant is
$$\sqrt{F^2 \cos^2\alpha + G^2 \sin^2\alpha}.$$

16.4 To resolve a given force into two perpendicular components

Let the line of action of the given force R make an angle θ with the direction OX and let OY be perpendicular to OX. Let P and Q be the resolved parts of R along OX and OY respectively.

Then $P = R \cos \theta$ and $Q = R \sin \theta$.

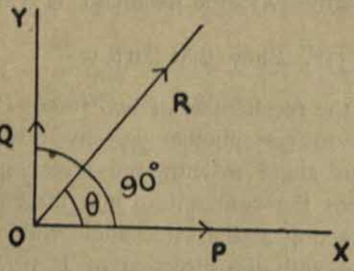


Fig. 16.3

REMARKS

1. The proof is similar to that in Section 11.6, Ch. 11, u, v, w being replaced by P, Q, R respectively.

2. The component of R along XO is $-R \,.\cos \theta$.

16.5 Theorem

The algebraic sum of the resolved parts of two forces along any direction is equal to the resolved part of their resultant in that direction.

EXAMPLE

Two forces R and S act at a point along two straight lines inclined at an angle θ, and F is their resultant. Two other forces R' and S' acting along the same lines have a resultant F'. If ϕ be the angle between the lines of action of F and F', prove that

$$(1-\cos\phi)(1+\cos\phi)$$
$$= \frac{(R^2S'^2 - 2RR'SS' + R'^2S^2)(1-\cos\theta)(1+\cos\theta)}{F^2F'^2}$$

Let OA and OB represent the two forces R and S, and OC their resultant F, so that $OACB$ is a parallelogram.

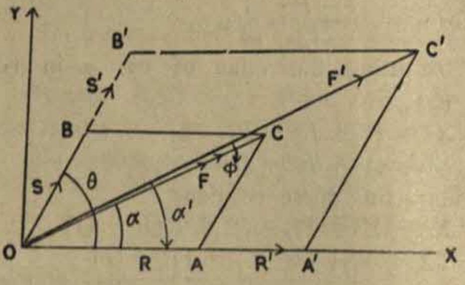


Fig. 16.4

Let OA' and OB' represent the two forces R' and S', and OC' their resultant F', so that $OA'C'B'$ is a parallelogram.

Let us take two directions OX and OY along and perpendicular to OA respectively. Resolving the forces R, S, F along OX, we have

$$F\cos\alpha = R + S\cos\theta$$

[cf. Section 16.5] (1)

Resolving the forces R, S, F along OY, we have

$$F\sin\alpha = S\sin\theta. \qquad (2)$$

Similarly, resolving R', S', F' along OX and OY, we have

$$F'\cos\alpha' = R' + S'\cos\theta \qquad (3)$$

and $$F'\sin\alpha' = S'\sin\theta. \qquad (4)$$

Here α, α' are the angles made by F, F' respectively with the line of action of R (i.e. of R').

$$\therefore\ \phi = \alpha' - \alpha.$$

$$\therefore\ \sin\phi = \sin(\alpha' - \alpha)$$
$$= \sin\alpha'\cos\alpha - \cos\alpha'\sin\alpha$$

$$\therefore\ FF'\sin\phi = F'\sin\alpha'.$$
$$F\cos\alpha - F'\cos\alpha' \,.\, F\sin\alpha$$
$$= S'\sin\theta\,(R + S\cos\theta)$$
$$-(R' + S'\cos\theta)\,S\sin\theta$$
$$= (RS' - R'S)\sin\theta$$

$$\therefore\ \sin^2\phi = \frac{(RS' - R'S)^2\sin^2\theta}{F^2F'^2}$$

$$\text{or,}\ (1-\cos^2\phi) = \frac{(RS' - R'S)^2(1-\cos^2\theta)}{F^2F'^2}$$

$$\therefore\ (1-\cos\phi)(1+\cos\phi) = \frac{(R^2S'^2 - 2RR'SS' + R'^2S^2)(1-\cos\theta)(1+\cos\theta)}{F^2F'^2}$$

Hence the result.

16.6 To find the resultant of a system of coplanar forces acting at a point

Let the coplanar forces P_1, P_2, P_3, ... acting at a point O be inclined at angles α_1, α_2, α_3, ... respectively with OX, OY being perpendicular to OX. Let the resultant R of the given system of forces be inclined at an angle θ with OX.

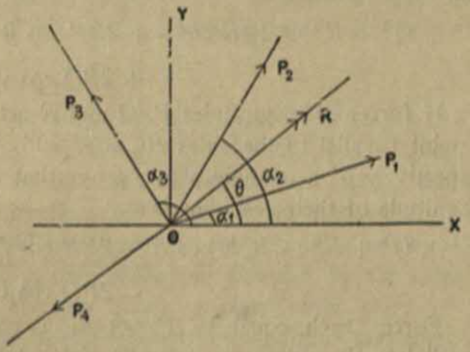


Fig. 16.5

We know by Section 16.5 that the algebraic sum of the resolved parts of a system of concurrent and coplanar forces in any direction is equal to the resolved part of their resultant in that direction.

Let X, Y be the resolved parts of the resultant R along OX and OY respectively. Then, resolving the forces along OX and OY, we get

$$X = R\cos\theta = P_1\cos\alpha_1 + P_2\cos\alpha_2 + P_3\cos\alpha_3 + \ldots = \Sigma_r P_r \cos\alpha_r \qquad (1)$$

and $$Y = R\sin\theta = P_1\sin\alpha_1 + P_2\sin\alpha_2 + P_3\sin\alpha_3 + \ldots = \Sigma_r P_r \sin\alpha_r. \qquad (2)$$

Squaring (1) and (2) and adding we have

$$R^2 = X^2 + Y^2 = (\Sigma_r P_r \cos\alpha_r)^2 + (\Sigma_r P_r \sin\alpha_r)^2$$

so that $$R = \sqrt{X^2+Y^2} \qquad (3)$$

Dividing (2) by (1), $\tan\theta = \dfrac{Y}{X} = \dfrac{\Sigma P_r \sin\alpha_r}{\Sigma P_r \cos\alpha_r}$

so that $$\theta = \tan^{-1}\frac{Y}{X}. \qquad (4)$$

Relation (3) gives the magnitude of the resultant and (4) gives its direction.

COROLLARY

1. *Conditions for equilibrium* : The given system of forces is said to be in equilibrium when the resultant $R = 0$.

Now $R = \sqrt{X^2+Y^2} = 0$ gives $X = 0$, $Y = 0$.

Hence the algebraic sum of the resolved parts of the forces along any two perpendicular directions should vanish separately when the forces are in equilibrium. This is the necessary condition for equilibrium.

Conversely, $X = 0$, $Y = 0$ means $R = 0$.

Hence the conditions $X = Y = 0$ are sufficient.

EXAMPLE

Forces act through the angular points of a triangle perpendicular to the opposite sides, and are proportional to the cosines of the corresponding angles ; show that their resultant is proportional to $\sqrt{1 - 8\cos A\cos B\cos C}$.

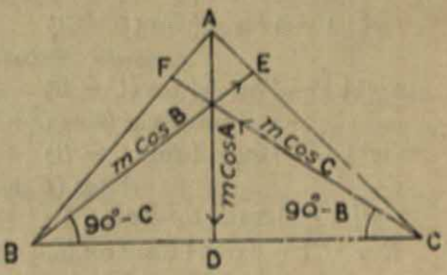


Fig. 16.6

Let AD, BE, CF be drawn perpendiculars from A, B, C respectively on the opposite sides.

Let the forces $m \cos A$, $m \cos B$ and $m \cos C$ act along AD, BE, CF respectively.

In $\triangle BEC$, $\angle BEC = 90°$ so that $\angle EBC = 90° - C$.

Similarly, $\angle BCF = 90° - B$.

Resolving the forces along and perpendicular to DC, we get

$$\begin{aligned} X &= m \cos B \cos (90° - C) \\ &\quad - m \cos C \cos (90° - B) \\ &= m \cos B \sin C - m \cos C \sin B \\ &= m \sin (C - B) \qquad (1) \end{aligned}$$

$$\begin{aligned} \text{and} \quad Y &= - m \cos A + m \cos B \sin (90° - C) \\ &\quad + m \cos C \sin (90° - B) \\ &= - m \cos A + m \cos B \cos C \\ &\quad + m \cos C \cos B \\ &= - m \cos A + 2m \cos B \cos C \\ &= - m \cos A + m \{ \cos (B + C) \\ &\quad + \cos (B - C) \} \\ &= - m \cos A + m \{ \cos (\pi - A) \\ &\quad + \cos (B - C) \} \\ &\qquad [\because \; A + B + C = \pi] \\ &= - m \cos A - m \cos A \\ &\quad + m \cos (B - C) \\ &= m \cos (B - C) - 2 m \cos A \qquad (2) \\ &= m \cos (C - B) - 2 m \cos A \end{aligned}$$

$\therefore$ The resultant R is given by

$$\begin{aligned} R^2 &= X^2 + Y^2 \\ &= m^2 \sin^2 (C - B) + m^2 \cos^2 (C - B) \\ &\quad - 4 m^2 \cos A \cos (C - B) + 4 m^2 \cos^2 A \\ &= m^2 [1 - 4 \cos A \{ \cos (C - B) \\ &\quad - \cos A \}] \\ &= m^2 [1 - 4 \cos A \{ \cos (C - B) \\ &\quad - \cos (\pi - \overline{C + B}) \}] \\ &= m^2 [1 - 4 \cos A \{ \cos (C - B) \\ &\quad + \cos (C + B) \}] \\ &= m^2 [1 - 8 \cos A \cos B \cos C] \end{aligned}$$

$$\therefore \quad R \propto \sqrt{1 - 8 \cos A \cos B \cos C}.$$

Exercise 16.2

1. Two forces P and Q have a resultant R and the resolved part of R in the direction of P is of magnitude Q. Show that the angle between the forces is $2 \sin^{-1}\left(\dfrac{P}{2Q}\right)^{\frac{1}{2}}$.

2. A particle is acted upon by forces represented by P, $2P$, $3\sqrt{3}P$ and $4P$; the angles between the first and second, the second and third, and the third and fourth are 60°, 90° and 150° respectively. Show that the resultant is a force P in a direction inclined at an angle of 120° to that of the first force.

3. Three forces P, Q, R in one plane act on a particle, the angles between Q and R, R and P, and P and Q being α, β, γ respectively. Show that their resultant is

$$\{P^2 + Q^2 + R^2 + 2QR \cos \alpha + 2RP \cos \beta + 2PQ \cos \gamma\}^{\frac{1}{2}}.$$

4. If forces of magnitudes P, Q and R act at a point parallel to the sides BC, CA and AB respectively of a triangle ABC, prove that the magnitude of their resultant is

$$\{P^2 + Q^2 + R^2 - 2QR \cos A - 2PR \cos B - 2PQ \cos C\}^{\frac{1}{2}}$$

5. Forces each equal to P act at a point parallel to the sides of a triangle ABC, taken in order ; show that their resultant is given by

$$P\{3 - 2 \cos A - 2 \cos B - 2 \cos C\}^{\frac{1}{2}}.$$

16.7 Triangle of forces

If three forces, acting at a point, be represented in magnitude and direction by the sides

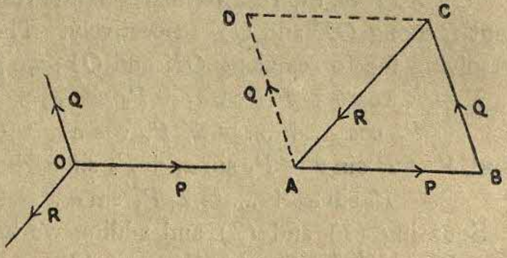


Fig. 16.7

of a triangle taken in order, they will be in equilibrium.

Let the three forces P, Q, R acting at O be represented, in magnitude and direction, by the sides AB, BC, CA respectively of $\triangle ABC$. Complete the parallelogram $ABCD$. Since AD is equal and parallel to BC, AD also represents the force Q. Hence the resultant of AB and BC is the same as the resultant of AB and AD. By parallelogram of forces, the resultant of AB and AD is AC.

Hence the resultant of P, Q, R
$\equiv$ resultant of AB, BC, CA
$\equiv$ resultant of AC and CA
$\equiv O$.
$\therefore$ P, Q, R are in equilibrium.

REMARKS

1. It should be noted that P, Q, R do not actually act along the sides of $\triangle ABC$; the sides of $\triangle ABC$ represent their magnitudes and directions only but not the positions.

2. If two forces acting at a point be represented in magnitude and direction by the sides of a triangle taken in order, then their resultant is represented in magnitude and direction (not in position) by the third side taken in opposite order. This may be symbolically stated as $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

16.8 Converse of the triangle of forces

If three forces acting at a point be in equilibrium, they can be represented in magnitude and direction by the sides of a triangle which may be drawn so as to have its sides respectively parallel to the direction of forces.

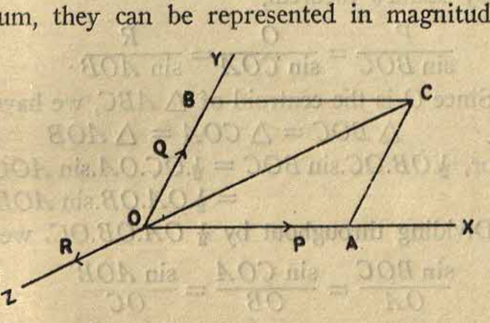


Fig. 16.8

Let the forces P, Q, R acting at O along OX, OY, OZ respectively, be in equilibrium. The lengths OA and OB are cut off to represent the magnitudes of P and Q respectively. The parallelogram $OACB$ is completed.

Since the three forces are in equilibrium, the resultant of any two of them must be equal and opposite to the third force.

By parallelogram of forces, the resultant of P and Q is represented by OC. Hence the third force R may be represented by CO. Also, AC being equal and parallel to OB, it represents Q. Hence the sides OA, AC, CO represent the forces P, Q, R respectively.

Now any triangle whose sides are parallel to those of $\triangle OAC$ will have its sides proportional to those of $\triangle OAC$ and hence proportional to the forces.

Again, any triangle whose sides are respectively perpendicular to those of $\triangle OAC$ will have its sides proportional to the sides of $\triangle OAC$ and hence proportional to the forces.

REMARK

Construction of a triangle of forces is possible unless the sum of two forces becomes less than the third.

16.9 Lami's theorem

If three forces acting at a point be in equilibrium, then each is proportional to the sine of the angle between the other two.

Let the forces P, Q, R acting at O along OX, OY, OZ respectively [see Fig. 16.8] be in equilibrium.

The lengths OA and OB are cut off from OX and OY respectively to represent the forces P and Q. The parallelogram $OACB$ is completed. Then CO represents R.

We have from Trigonometry,

$$\frac{OA}{\sin ACO}=\frac{AC}{\sin COA}=\frac{CO}{\sin OAC}$$

$$\begin{aligned}
\text{Now } \sin\angle ACO &= \sin\angle BOC\\
&= \sin(180^\circ - YOZ)\\
&= \sin YOZ = \sin(Q,\ R),\\
\sin COA &= \sin(180^\circ - ZOX)\\
&= \sin ZOX = \sin(R,\ P),\\
\sin OAC &= \sin(180^\circ - CAX)\\
&= \sin CAX = \sin YOX\\
&= \sin(P,\ Q).
\end{aligned}$$

$$\therefore\quad \frac{OA}{\sin(Q,R)}=\frac{OB}{\sin(R,P)}=\frac{CO}{\sin(P,Q)}\qquad [\because AC = OB]$$

$$\text{or,}\quad \frac{P}{\sin(Q,R)}=\frac{Q}{\sin(R,P)}=\frac{R}{\sin(P,Q)}$$

Here sin (Q, R) denotes sine of the angle between Q and R.

16.10 Illustrative examples

EXAMPLE 1

If P be any point in the plane of the triangle ABC and D, E, F be the middle points of its sides BC, CA and AB respectively, show that the forces AP, BP, CP, PD, PE, PF are in equilibrium.

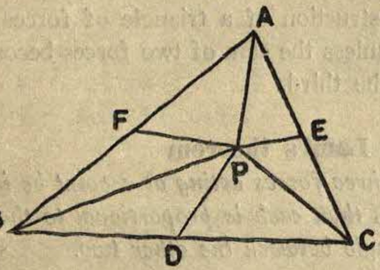


Fig. 16.9

The forces PA and PB are equivalent to $2PF$; (by using the $\lambda - \mu$ theorem) the forces PB and PC are equivalent to $2PD$ and the forces PC and PA are equivalent to $2PE$.

Hence the sum of the forces represented by $2PF$, $2PD$, $2PE$ is equivalent to the sum of the forces represented by $2PA$, $2PB$, $2PC$; in other words, the sum of forces PA, PB, PC is equivalent to the sum of the forces PD, PE, PF.

Therefore, the sum of the forces AP, BP, CP, PD, PE, PF is equivalent to the sum of the forces AP, BP, CP, PA, PB, PC. But the constituents of the latter sum balance each other and hence the forces AP, BP, CP, PD, PE, PF maintain equilibrium.

EXAMPLE 2

ABC is a triangle. Forces P, Q, R act along the lines OA, OB and OC and are in equilibrium. If O be the centroid of the triangle ABC, prove that

$$\frac{P}{OA}=\frac{Q}{OB}=\frac{R}{OC}.$$

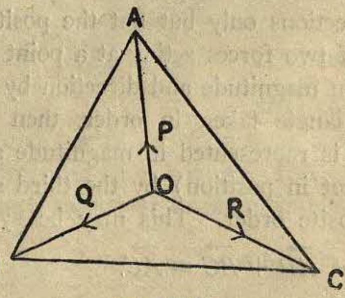


Fig. 16.10

By Lami's theorem,

$$\frac{P}{\sin BOC}=\frac{Q}{\sin COA}=\frac{R}{\sin AOB}\qquad (1)$$

Since O is the centroid of $\triangle ABC$, we have

$$\triangle BOC = \triangle COA = \triangle AOB$$

$$\begin{aligned}
\text{or, } \tfrac{1}{2}.OB.OC.\sin BOC &= \tfrac{1}{2}.OC.OA.\sin AOC\\
&= \tfrac{1}{2}.OA.OB.\sin AOB.
\end{aligned}$$

Dividing throughout by $\frac{1}{2}\ OA.OB.OC$, we get

$$\frac{\sin BOC}{OA}=\frac{\sin COA}{OB}=\frac{\sin AOB}{OC}\qquad (2)$$

By (1) and (2), we get $\dfrac{P}{OA}=\dfrac{Q}{OB}=\dfrac{R}{OC}.$

Exercise 16.3

1. Determine whether a particle can be kept at rest by three simultaneous forces whose magnitudes are proportional to (i) 4, 5, 9, (ii) 12, 3, 8, (iii) 4, 7, 9.

2. ABC is a triangle and D, E, F are the middle points of the sides BC, CA, AB respectively. Forces represented by AD, $\frac{2}{3}BE$ and $\frac{1}{3}CF$ act on a particle. Show that their resultant is represented by $\frac{1}{2}AC$ and that its line of action divides BC in the ratio 2 : 1.

3. The sides AB and AC of a triangle ABC are bisected in D and E; show that the resultant of forces represented by BE and DC is represented in magnitude and direction by $\frac{3}{2}BC$.

4. D, E, F are the middle points of the sides BC, CA, AB respectively. Show that the forces acting on a particle and represented by the straight lines AD, BE, CF will maintain equilibrium.

5. Show that the system of forces represented by the lines joining any point to the angular points of a triangle is equivalent to the system represented by straight lines drawn from the same point to the middle points of the sides of the triangle.

6. Forces acting at a point are represented in magnitude, direction and order by AB, $2BC$, $2CD$, DA, DB where $ABCD$ is a square. Show that the forces are in equilibrium.

7. Find a point within a quadrilateral such that, if it be acted on by forces represented by the lines joining it to the angular points of the quadrilateral, it will be in equilibrium.

8. The sides BC and DA of a quadrilateral $ABCD$ are bisected in E and F respectively. Show that if two forces parallel and equal to AB and DC act on a particle, their resultant is parallel to FE and equal to $2FE$.

9. Forces acting at a point are represented in magnitude and direction by $2AB$, $3BC$, $2CD$, DA, CA and DB where $ABCD$ is a quadrilateral. Show that the forces are in equilibrium.

10. A body of mass 10 lbs. is suspended by two strings 7 and 24 inches long, their other ends being fastened to the extremities of a rod of length 25 inches. If the rod be so held that the body hangs immediately below its middle point, find the tensions in the strings.

11. A string ACB has its extremities tied to two fixed points A and B in the same horizontal line; to a given point C is knotted a given weight W. Prove that the tension in the portion CA is $\frac{Wb}{4c\triangle}(c^2 + a^2 - b^2)$ where a, b, c are the sides and $\triangle$ the area of the triangle ABC.

17 Parallel Forces

In the previous chapter, we have considered the composition and resolution of concurrent forces. In the present chapter, we shall deal with coplanar parallel forces acting upon a rigid body.

When the lines of action of two forces do not meet at a point, they are said to be parallel and are said to be **like** when they act in the same direction and **unlike** when they act in the opposite directions.

17.1 Resultant of two like parallel forces

Let the two like parallel forces P and Q, acting at the points A and B of a rigid body, be represented by AL and BG respectively. We introduce, at the points A and B, two equal and opposite forces F represented by AN and BK. These forces being equal and opposite do not affect the resultant of P and Q. The parallelograms $ALMN$ and $BGHK$ are completed. By parallelogram of forces, the resultant R of P and F at A is represented by AM, and the resultant R' of Q and F at B is represented by BH. MA and HB are produced to meet at D. Through D, DC is drawn parallel to P and Q to meet AB at C. The points of application of R and R' may be transferred to D. The force R is now resolved into its initial components P and F acting along DC and parallel to AB respectively. Similarly, R' can be resolved into components Q and F acting along DC and parallel to AB.

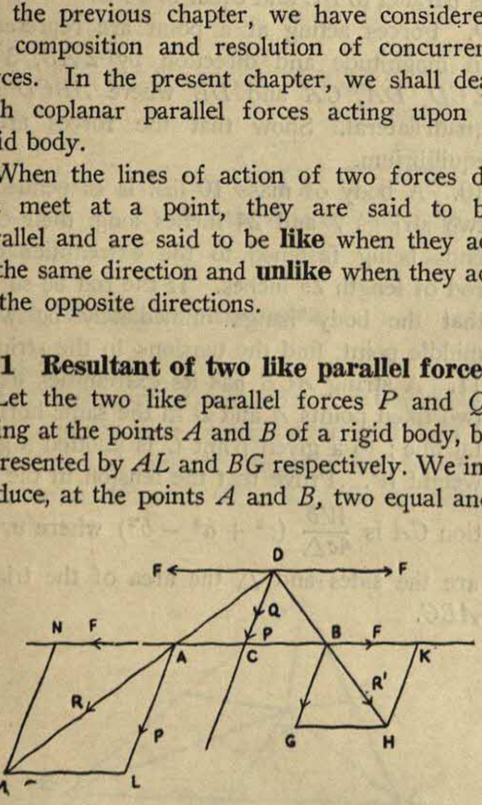


Fig. 17.1

The two forces F, F at D balance each other while the forces P and Q acting along DC have the resultant $(P + Q)$ acting along DC.

From the similar triangles ACD and MLA, we have

$$\frac{DC}{AC} = \frac{AL}{ML} = \frac{AL}{AN} = \frac{P}{F}$$

$$\text{or,}\quad F \,.\, DC = P \,.\, AC. \qquad (1)$$

Similarly, from the similar triangles BCD and BGH,

$$F \,.\, DC = Q \,.\, BC. \qquad (2)$$

From (1) and (2), $P \,.\, AC = Q \,.\, BC$

$$\text{or,}\quad \frac{AC}{BC} = \frac{Q}{P}.$$

Hence C divides AB internally in the inverse ratio of the forces.

17.2 Resultant of two unlike unequal parallel forces

Let P and Q $(Q > P)$ be two unlike parallel forces acting at the points A and B respectively of a rigid body. Let us introduce a force F (in the direction of P) at C on AB produced—the position and the magnitude of F being so chosen as to produce a resultant Q at B in the direction of P. Another force F is now introduced at C in a direction opposite to the former one.

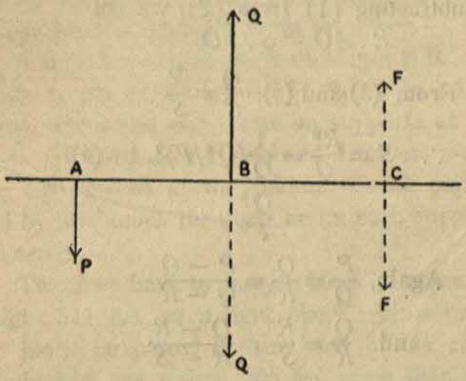


Fig. 17.2

The two forces F, F at C being equal and opposite, balance each other and hence they do not affect the resultant of P and Q.

By construction and assumption,

$$Q = P + F \qquad (1)$$

$$\text{and} \quad P \,.\, AB = F \,.\, BC. \qquad (2)$$

Now the two forces Q at B balance each other and we are left with a single force F at C in the direction of Q.

Hence we may conclude that the resultant of P at A and Q at B is $F = Q - P$ acting at C in the direction of the greater force Q.

From (2) & (1),

$$P \,.\, AB = (Q - P) \,.\, BC$$

$$\text{or,} \quad P(AB + BC) = Q \,.\, BC$$

$$\text{or,} \quad P \,.\, AC = Q \,.\, BC \qquad (3)$$

$$\text{or,} \quad \frac{P}{Q} = \frac{BC}{AC}$$

Hence the line of action of the resultant divides AB externally in the inverse ratio of the forces.

REMARK

The method of composition of two **equal unlike parallel forces** will be discussed in a subsequent chapter.

COROLLARY

Conditions of equilibrium of three parallel forces: If three parallel forces are in equilibrium, each force is proportional to the distance between the other two.

If R be the resultant of two like parallel forces P and Q acting at A and B respectively and if its line of action intersects AB at C, we have from Section 17.1 above, $P \,.\, AC = Q \,.\, BC$

$$\text{so that} \quad \frac{P}{BC} = \frac{Q}{AC} = \frac{P+Q}{BC+AC} = \frac{R}{AB} \qquad \text{(i)}$$

Similarly, when P and Q are unlike unequal parallel forces, we have from Section 17.2,

$$\frac{P}{BC} = \frac{Q}{AC} = \frac{Q-P}{AC-BC} = \frac{R}{AB} \qquad \text{(ii)}$$

Relations (i) and (ii) establish the result.

REMARKS

1. The sum of the resolved parts of two like or unlike (unequal) parallel forces in any direction is equal to the resolved part of their resultant in that direction.
2. The resultant of any number of parallel forces is parallel to them and equal to their algebraic sum.

17.3 Illustrative examples

EXAMPLE 1

Two men, one stronger than the other, have to remove a block of stone 500 kgs., with a light pole whose length is 250 cms; the weaker man cannot carry more than 200 kgs. Where must

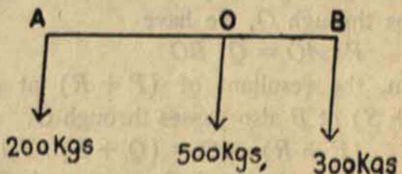


Fig. 17.3

the stone be fastened to the pole so as just to allow him his full share of weight ?

Let A and B be the positions of the weaker and the stronger man respectively. When the man at A carries 200 kgs., the man at B must carry the remaining 300 kgs. Let O be the position of the stone. The weight at O is balanced by the weights at A and B.

$$\therefore\ 200\,.\,AO = 300\,.\,BO$$

$$\text{or,}\ 2\,.\,AO = 3\,.\,BO$$

$$\text{or,}\ \frac{AO}{3} = \frac{BO}{2} = \frac{AO + BO}{3 + 2} = \frac{AB}{5} = \frac{250}{5} = 50$$

$\therefore$ $AO = 150$ cms. and $BO = 100$ cms.

$\therefore$ The stone is to be fastened to the pole at a distance of 150 cms. from the weaker man.

EXAMPLE 2

The resultant of two like parallel forces P, Q passes through a point O ; when P is increased by R and Q by S, the resultant still passes through O, and also when Q, R replace P, Q respectively.

Show that $S = \dfrac{Q^3}{P^2} = R - \dfrac{(Q-R)^2}{P-Q}$.

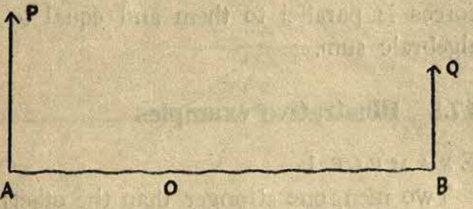


Fig. 17.4

Let the two like parallel forces P and Q act at A and B respectively. Since their resultant passes through O, we have

$$P\,.\,AO = Q\,.\,BO \qquad (1)$$

Again, the resultant of $(P+R)$ at A and $(Q+S)$ at B also passes through O.

$$\therefore\ (P+R)\,.\,AO = (Q+S)\,.\,BO \qquad (2)$$

Also, the resultant of Q at A and R at B passes through O.

$$\therefore\ Q\,.\,AO = R\,.\,BO \qquad (3)$$

From (1) & (3),

$$\frac{BO}{AO} = \frac{P}{Q} = \frac{Q}{R}$$

$$\text{or,}\quad R = \frac{Q^2}{P} \qquad (4)$$

Subtracting (1) from (2), we get

$$R\,.\,AO = S\,.\,BO \qquad (5)$$

$$\text{From (3) and (5),}\ \frac{Q}{R} = \frac{R}{S} \qquad (6)$$

$$\therefore\ S = \frac{R^2}{Q} = \frac{1}{Q}(Q^2/P)^2 \ \text{by (4)}$$

$$= \frac{Q^3}{P^2} \qquad (7)$$

$$\text{Again,}\ \frac{P}{Q} = \frac{Q}{R} = \frac{P-Q}{Q-R}$$

$$\text{and}\ \frac{Q}{R} = \frac{R}{S} = \frac{Q-R}{R-S}.$$

$$\therefore\ \frac{P-Q}{Q-R} = \frac{Q-R}{R-S}$$

$$\text{or,}\ R - S = \frac{(Q-R)^2}{P-Q}$$

$$\therefore\ S = R - \frac{(Q-R)^2}{P-Q} \qquad (8)$$

By (7) and (8), $S = \dfrac{Q^3}{P^2} = R - \dfrac{(Q-R)^2}{P-Q}$.

Hence the result.

Exercise 17.1

1. Find two like parallel forces acting at a distance of 75 cms. apart, which are equivalent to a given force of 50 kgs. wt., the line of action of one being at a distance of 15 cms. from the given force.
2. Find two unlike parallel forces acting at a distance of 18 inches apart which are equivalent to a force of 30 lbs. wt., the greater of the two forces being at a distance of 8 inches from the given force.
3. The extremities of a straight bamboo pole 8 ft. long rest on two smooth pegs P and Q in the same horizontal line. A heavy load hangs

from a point R of the pole. If $PR = 3\,RQ$, and the pressure at Q be 325 lbs. more than that at P, find the weight of the load.

4. A uniform beam, 4 ft. long, is supported in a horizontal position by two props which are 3 ft. apart, so that the beam projects one foot beyond one of the props. Show that the force on one prop is double that on the other.

5. A light horizontal plank of length 8 ft., on which is placed a load of 32 lbs. at a point 1 foot from one end, rests on supports at its ends. If the load be removed from its position and placed at the middle of the plank, find by how much the pressure on each support is altered.

6. Two men have to carry a block of stone of weight 311 lbs. on a light plank. How must the block be placed so that one of the men should bear the weight 205 lbs. more than the other ?

7. Two like parallel forces P and Q act at given points of a body ; if Q be changed to P^2/Q, show that the line of action of the resultant is the same as it would be if the forces were simply interchanged.

8. If the magnitudes of two unlike parallel forces P, Q $(P > Q)$ be increased by the same amount, show that the line of action of the resultant will move further off from P.

9. P and Q are two like parallel forces acting at A and B respectively. Show that if they interchange position, the point of application of the resultant is displaced through a distance

$$\frac{P-Q}{P+Q} \cdot AB \; (P > Q) \text{ along } AB.$$

Hence or otherwise prove that if the position of the resultant remains unaltered when the forces are interchanged, the forces must be equal in magnitude.

10. A man carries a bundle at the end of a stick which is placed horizontally over his shoulder ; if the distance between his hand and his shoulder be changed, how does the pressure on his shoulder change ?

11. A rod, of length $2l$ and weight W, is lying across two pegs on the same level a feet apart. If neither peg can bear a pressure greater than P, find the greatest length of the rod which may project beyond either peg.

12. P, Q are like parallel forces. If P is moved parallel to itself through a distance d, find the distance through which the resultant of P and Q moves.

18 Moment of a Force

Let us suppose that a body is attached to a fixed axis or hinge about which it can rotate freely. The body can be set in motion by applying to it a force in any direction not passing through the axis. It has been verified by experiments that this tendency of rotation varies as the magnitude of the force and also as the perpendicular distance of the force from the axis of rotation. Actually, the further off from the axis the force is applied, the more effect it has in turning the body.

18.1 Definition of moment of a force about a point

The moment of a force about a fixed point is the product of the magnitude of the force and the perpendicular distance of the point from its line of action.

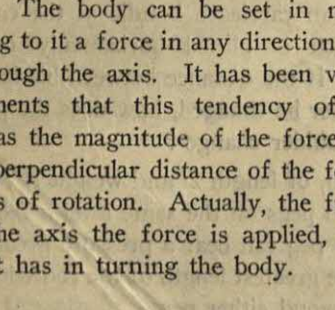


Fig. 18.1

Thus the moment of a force P about a point O in its plane is the product $P \times OM$ where OM is the perpendicular distance of the line of action of P from O.

The length OM is called the **arm of the moment.**

Hence *moment=force×arm.*

REMARKS

1. The moment $P \times OM$ vanishes if either $P=0$ or $OM=0$. Here $OM=0$ implies that the point O lies on the line of action of P.

Hence we conclude that the moment of a force vanishes when either (i) the force itself is zero, or (ii) the line of action of the force passes through the point.

2. It has been verified by experiments that the moment of a force about any point is a proper measure of the tendency of the force to produce rotation about that point.

3. *Positive and negative moments*: The moment of a force about a point is regarded as positive or negative according as the force tends to rotate a body about that point in the anticlockwise or clockwise direction.

It is convenient to remember that the moment of a force is

(i) **positive** about all points on the **left** of its line of action, and

(ii) **negative** about all points on the **right** of its line of action,

with regard to a person looking in the direction of the force.

18.2 Geometrical representation of the moment of a force

Let AB represent the force P in magnitude and direction, and O any point in its plane. OM is drawn perpendicular to AB.

Now, moment of P about O

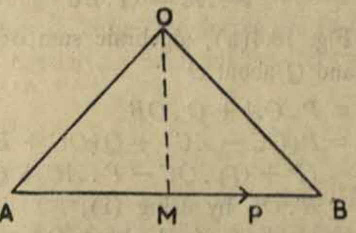


Fig. 18.2

$$
\begin{aligned}
&= P \times OM = AB \times OM\\
&\qquad\qquad = 2\cdot\tfrac{1}{2}AB \times OM\\
&\qquad\qquad = 2 \times \text{area of } \triangle OAB.
\end{aligned}
$$

Hence we may conclude :

If a force be completely represented by a line segment, its moment about any point is measured by twice the area of the triangle whose base is the line segment and whose vertex is the point about which moment is taken.

COROLLARY

1. If a given force is compounded with any force which passes through O, the moment about O remains unaltered.

2. If a force is applied at any point A, its moment about any point O is

$OA \times$ resolved part of P perpendicular to OA.

We have from Fig. 18.2, moment of P about O

$$
\begin{aligned}
&= P \times OM = P \times OA \sin OAM\\
&\qquad\qquad = OA \times P \sin OAM\\
&\qquad\qquad = OA \times \text{resolved part of } P \text{ perpendicular to } OA.
\end{aligned}
$$

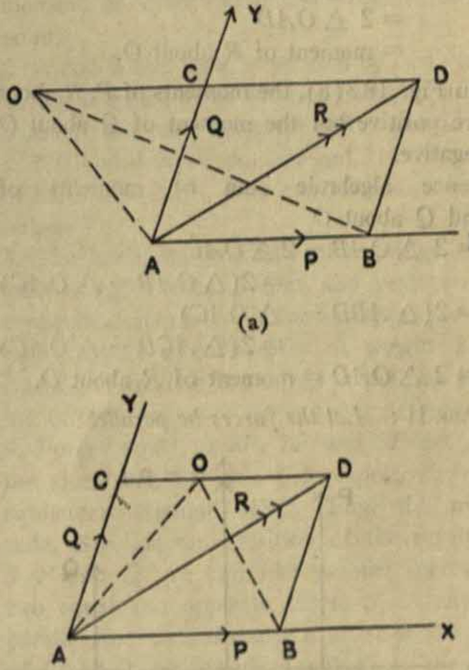
3. The moments of two intersecting forces about any point in the line of action of their resultant are equal and opposite.

18.3 Varignon's theorem

The algebraic sum of the moments of two forces about any point in their plane is equal to the moment of their resultant about that point.

We shall consider two cases as proofs of the above theorem.

CASE I : *Let the forces meet at a point :*



(a)

(b)

Fig. 18.3

Let the two forces P and Q act at the point A along AX and AY respectively, and let the moments be taken about the point O which may lie within the angle XAY [see Fig. 18.3 (b)] or outside it [see Fig. 18.3(a)].

The line OC is drawn through O parallel to P to meet the line of action of Q at C. Let us choose a scale such that AC represents the magnitude of Q, and on the same scale let AB represent the magnitude of P.

The parallelogram $ABDC$ is completed. Then the diagonal AD represents the resultant R of P and Q. OA, OB, AD are joined.

In Fig. 18.3(a), the moments of both P and Q about O are positive. Hence algebraic sum of moments of P and Q about O

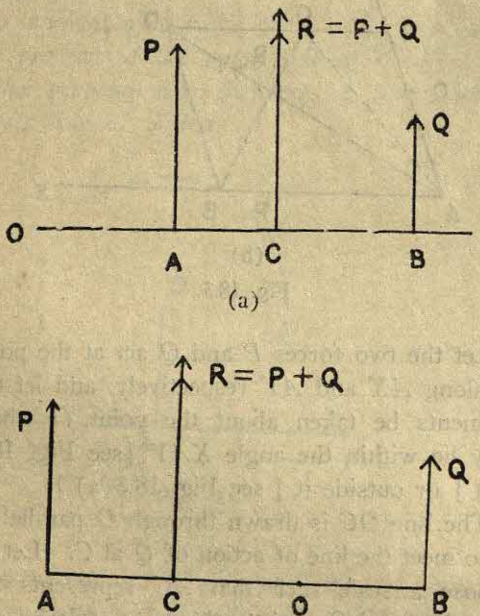
$$
\begin{aligned}
&= 2\ \triangle OAB + 2\ \triangle OAC \\
&= 2(\triangle ABD + \triangle OAC) \\
&= 2(\triangle ADC + \triangle OAC) \\
&= 2\ \triangle OAD \\
&= \text{moment of } R \text{ about } O.
\end{aligned}
$$

In Fig. 18.3(b), the moments of P, R about O are positive but the moment of Q about O is negative.

Hence algebraic sum of moments of P and Q about O

$$
\begin{aligned}
&= 2\ \triangle OAB - 2\ \triangle OAC \\
&\qquad = 2(\triangle OAB - \triangle OAC) \\
&= 2(\triangle ABD - \triangle OAC) \\
&\qquad = 2(\triangle ACD - \triangle OAC) \\
&= 2\ \triangle OAD = \text{moment of } R \text{ about } O.
\end{aligned}
$$

CASE II : *Let the forces be parallel* :



(a)

(b)

Fig. 18.4

Let P and Q be two like parallel forces and let O be any point in their plane. A line is drawn through O perpendicular to the lines of action of the forces P and Q to meet them at A and B respectively. Then the resultant $R = P + Q$ acts at C on AB such that

$$P \, . \, AC = Q \, . \, BC \qquad \text{(i)}$$

In Fig. 18.4(a), algebraic sum of moments of P and Q about O

$$
\begin{aligned}
&= P \, . \, OA + Q \, . \, OB \\
&= P(OC - AC) + Q(OC + BC) \\
&= (P + Q) \, . \, OC - P \, . \, AC + Q \, . \, BC \\
&= R \, . \, OC \text{ by using (i),} \\
&= \text{moment of } R \text{ about } O.
\end{aligned}
$$

In Fig. 18.4(b), algebraic sum of moments of P and Q about O

$$
\begin{aligned}
&= - P \, . \, OA + Q \, . \, OB \\
&= - P(AC + OC) + Q(BC - OC) \\
&= - (P + Q) \, . \, OC - P \, . \, AC + Q \, . \, BC \\
&= - R \, . \, OC \text{ by using (i),} \\
&= \text{moment of } R \text{ about } O.
\end{aligned}
$$

Hence the theorem.

REMARK

In case the parallel forces are unlike and unequal, we can similarly prove the theorem.

COROLLARY

The algebraic sum of the moments of any two coplanar forces (which are not in equilibrium) about any point which lies on the line of action of their resultant is zero, and conversely.

18.4 Generalisation of Varignon's theorem

Varignon's Theorem on two forces may be easily generalised to the case of any number of coplanar forces as stated below :

The algebraic sum of the moments of any number of coplanar forces about any point in their plane is equal to the moment of their resultant about that point.

COROLLARY

If a given system of coplanar forces be in equilibrium, the algebraic sum of their moments about any point in the plane is zero.

If the algebraic sum of the moments of a number of coplanar forces about any point in their plane be zero, then either their resultant passes through that point or the resultant itself is zero; that is, either the forces are in equilibrium or they have a resultant passing through the point.

EXAMPLE

A heavy carriage wheel of weight W and radius r is to be dragged over an obstacle of height h, by a horizontal force P applied to the centre of the wheel. Show that P must be slightly greater than $W \dfrac{\sqrt{2rh - h^2}}{(r - h)}$.

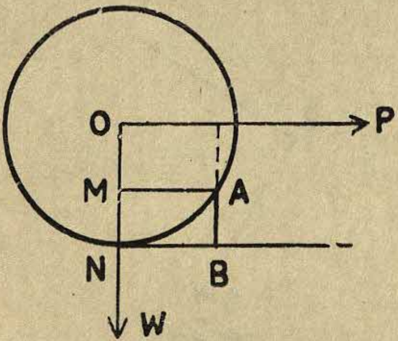


Fig. 18.5

Let O be the centre of the wheel and AB the obstacle of height h. It is possible to drag the wheel over the obstacle if the moment of the force P about A be slightly greater than the moment of the weight W about A.

i.e. if $P \, . \, OM > W \, . \, AM$

Now $OM = ON - MN = r - AB = r - h$

and $AM = \sqrt{OA^2 - OM^2} = \sqrt{r^2 - (r-h)^2}$

$= \sqrt{2rh - h^2}$.

Hence $P > W \cdot \dfrac{AM}{OM}$

i.e. $P > \dfrac{W\sqrt{2rh - h^2}}{(r-h)}$.

Exercise 18.1

1. ABC is a triangle, right angled at C, and having the angle B equal to 60°. Forces 3, 4, 5 act along AB, BC, CA respectively. Find the moment of each round the opposite angular point.
2. A rod, 5 feet long, supported by two vertical strings attached to its ends, has weights of 4, 6, 8 and 10 lbs. hung from the rod at distances of 1, 2, 3 and 4 feet from one end. If the weight of the rod be 2 lbs., what are the tensions of the strings?
3. The horizontal roadway of a bridge AB is 36 ft. long, weighs 5 tons and rests on two supports at its ends. What is the pressure on each support when a lorry of weight 3 tons starting from A is two-thirds of the way across the bridge?
4. Forces equal to $3P$, $7P$ and $5P$ act along the sides AB, BC and CA respectively of an equilateral triangle ABC. Find the magnitude, direction and position of the resultant.
5. P and Q are two like parallel forces. If two equal and opposite forces S, in any two parallel lines at a distance d apart in the plane of P and Q, are combined with them, show that their resultant is displaced by a distance
$$\frac{dS}{P+Q}.$$
6. Forces P, Q, R act along the sides BC, CA, AB of a triangle. The line of action of their resultant divides BC in the ratio $k : l$ and AC in the ratio $h : l$. Prove that $P : Q : R = ha : kb : (-lc)$.
7. AOB is a weightless rod bent to form an angle α at O. Weights P and Q are attached at A and B, and the whole system is suspended

from O. Given $OA = a$, $OB = b$, prove that if θ be the inclination of OA to the horizontal,

$$\tan\theta = \frac{Pa}{Qb\sin\alpha} + \cot\alpha.$$

8. *A* uniform plank of length $2a$ and weight W is supported horizontally on two vertical props at a distance b apart. The greatest weight that can be placed at the two ends in succession without upsetting the plank are W_1 and W_2 respectively. Show that

$$\frac{W_1}{W+W_1} + \frac{W_2}{W+W_2} = \frac{b}{a}.$$

9. *A* narrow uniform plank 20 ft. long weighing 100 lbs. is supported in a horizontal position on two posts, one 5 ft. from one end, and the other 8 ft. from the other end of the plank. A boy weighing 60 lbs. walks on it starting from the latter post towards the corresponding end. Find how far it is safe for him to walk. What are the reactions of the posts when he is farthest from the starting point without upsetting the plank ?

10. At what height from the base of a pillar must the end of a rope of given length be fixed so that a man standing on the ground and pulling at its other end with a given force may have the greatest tendency to make the pillar overturn ?

19 Couples

19.1 Definition

Two equal forces acting in opposite directions along two parallel straight lines are said to form a **couple.**

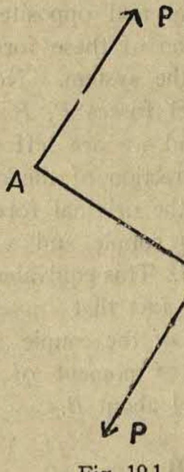


Fig. 19.1

The **arm** of a couple is the perpendicular distance between the lines of action of the two forces forming the couple.

The **moment** of a couple is the algebraic sum of the moments of the two forces forming the couple about any point in their plane.

Fig. 19.1 shows a couple, each of whose forces is P and whose arm is p; the couple is usually denoted by (P, p).

A couple cannot keep a body in equilibrium because it tends to rotate the body, the points of application of the two forces of the couple tending to move in opposite directions.

Examples of a couple are the forces applied to the key of a lock, to the handle of a door in opening it, to a small top in spinning it.

19.2 A fundamental property of couples

The moment of a couple is the same about all points in its plane.

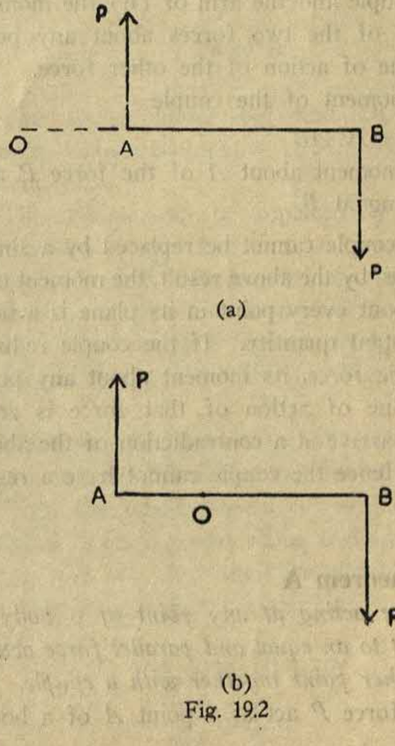


Fig. 19.2

Let the couple consist of two equal unlike parallel forces P, P. Let O be any point in the plane of the couple. OAB is drawn perpendicular to the lines of action of the forces.

Then, if O does not lie between A and B as in Fig. 19.2(a), moment of the couple about O = algebraic sum of moments of forces about O

$$= P\,.\,OA - P\,.\,OB = -P(OB - OA)$$
$$= -P\,.\,AB.$$

When O lies between A and B as in Fig. 19.2(b), moment of the couple about O = algebraic sum of moments about O

$$= -P\,.\,OA - P\,.\,OB = -P(OA + OB)$$
$$= -P\,.\,AB.$$

Hence the moment of the couple about O is independent of the position of O and is equal to the product $P\,.\,AB$ in magnitude.

REMARKS

1. The moment of a couple may therefore be defined as (i) the product of either force of the couple and the arm or (ii) the moment of either of the two forces about any point in the line of action of the other force.

For, moment of the couple

$= -P\,.\,AB$

= moment about A of the force P acting at B.

2. A couple cannot be replaced by a single force. For, by the above result, the moment of a couple about every point in its plane is a non-zero constant quantity. If the couple reduces to a single force, its moment about any point on the line of action of that force is zero. We thus arrive at a contradiction of the above result. Hence the couple cannot have a resultant.

19.3 Theorem A

A force acting at any point of a body is equivalent to an equal and parallel force acting at any other point together with a couple.

Let a force P act at a point A of a body. AB is drawn perpendicular to the line of action of P. At some point B on this perpendicular, let us introduce two equal and opposite forces each equal and parallel to P at A. These two forces being equal and opposite cancel each other. Introduction of these forces does not, therefore, affect the system. Now the equal and unlike parallel forces P, P at A and B form a couple and we are left with a force P at B in the direction of the original force P at A. Hence the original force P at A is equivalent to this couple and a parallel and equal force P at A. This equivalence is further established by the fact that

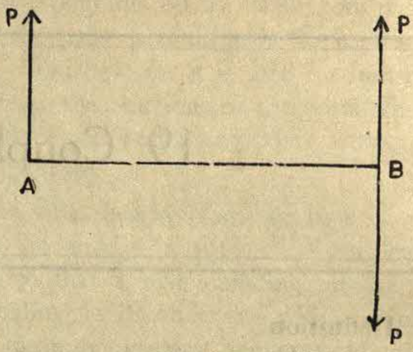


Fig. 19.3

the moment of the couple
$= -P\,.\,AB =$ moment of the original force P at A about B.

19.4 Theorem B

Two couples in the same plane whose moments are equal and opposite will balance each other.

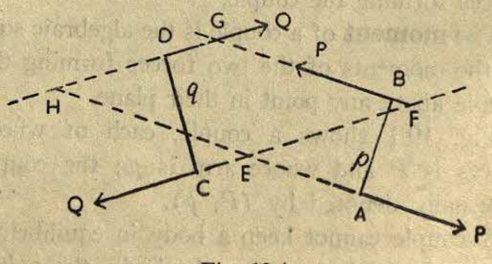


Fig. 19.4

COROLLARY

A couple acting on a rigid body may be replaced, without altering its effect, by any other couple of equal moment acting on the body in the same plane.

19.5 Theorem C

A single force and a couple acting in the same plane are equivalent to the given single force acting in a direction parallel to its original direction.

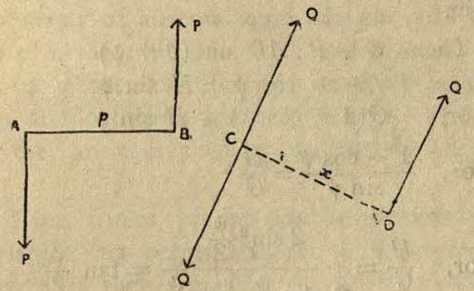


Fig. 19.5

Let the given couple consist of the two forces P, P acting on the arm $AB = p$ and let Q be the given single force acting at C.

The couple (P, p) may be replaced by a couple of equal moment in its plane. Let us replace the given couple by a couple (Q, x) such that $Pp = Qx$. One of the constituent forces of the couple (Q, x) is taken at C in a direction opposite to that of the given force at C and the other constituent force Q is taken at D where $CD (= x)$ is perpendicular to the line of action of the given force Q at C.

The two equal and opposite forces Q, Q at C balance each other and we are left with a single force Q at D which is parallel to the given force Q.

Thus the couple (P, p) and the force Q at C reduce to the single force Q which acts in a direction parallel to the given force Q at C but displaced through a distance $x = \frac{Pp}{Q}$.

REMARK

It is evident from this theorem that a force and a couple in the same plane cannot produce equilibrium.

19.6 Illustrative examples

EXAMPLE 1

ABCD is a rectangle whose sides AB, BC measure 8 and 3 cms. respectively. Forces of 16, 5, 16, 9 gm. wt. act along the sides BA, BC, DC, DA respectively. Find their resultant.

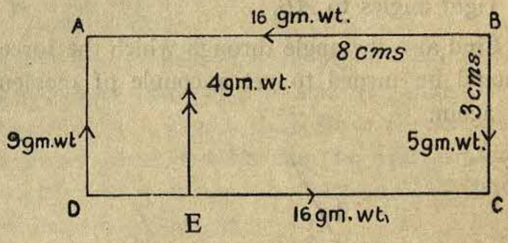


Fig. 19.6

The forces along BA and DC form a couple whose moment $= 16 \times 3$ cm.-gms. $=$ 48 cm.-gms.

This couple can be replaced by a couple consisting of two forces 6, 6 gm.wt. acting along AD and CB respectively because the moment of this couple is also 48 cm.-gms.

Now the forces 9 gm.wt. along DA and 6 gm.wt along AD reduce to 3 gm.wt. along DA.

Similarly, the forces 5 gm.wt. along BC and 6 gm.wt along CB reduce to 1 gm.wt. along CB.

Thus the whole system reduces to two like parallel forces 3 gm.wt. along DA and 1 gm.wt. along CB, so that their resultant is a like parallel force 4 gm.wt. acting at E on DC such that

$$3 . DE = 1 . CE$$

$$\text{or,} \quad \frac{DE}{CE} = \frac{1}{3}.$$

Since $CD = 8$ cms., $DE = 2$ cms.

Hence the resultant of the system is a force 4 gm.wt. parallel to DA acting at a point E on DC at a distance of 2 cms. to the right of D.

EXAMPLE 2

The constituent forces of a couple of moment G act at A and B ; if their lines of action are turned through a right angle, they form a couple of moment H. Show that they form a couple of moment (i) $G \cos \alpha + H \sin \alpha$ when each of them is turned through an angle α ; (ii) $\sqrt{G^2 + H^2}$ when they both act at right angles to AB.

Find also the angle through which the forces should be turned to get a couple of moment G again.

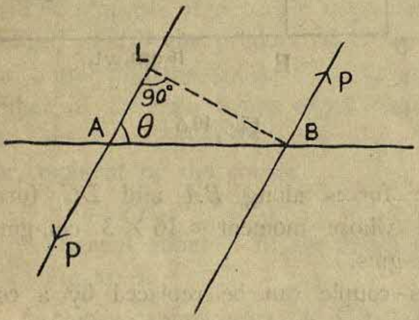


Fig. 19.7

Let the couple of moment G consist of forces P, P acting at A and B at an angle θ with AB as shown in Fig. 19.7.

Then $G = P \,.\, BL = P \,.\, AB \sin \theta$ (1)

Also $H = P \,.\, AB \sin (90° + \theta)$

$= P \,.\, AB \cos \theta$ (2)

Squaring (1) and (2) and then adding together,

$$P \,.\, AB = \sqrt{G^2 + H^2}. \qquad (3)$$

(i) Let G' be the moment of the couple when each force is turned through an angle α.

Then

$$\begin{aligned} G' &= P \,.\, AB \sin (\theta + \alpha) \\ &= P \,.\, AB \sin \theta \cos \alpha + P \,.\, AB \cos \theta \sin \alpha \\ &= G \cos \alpha + H \sin \alpha \text{ by (1) \& (2).} \end{aligned}$$

(ii) When the forces act at right angles to AB, we have $\theta = 90°$ so that

the moment of the couple $= P \,.\, AB$

$= \sqrt{G^2 + H^2}$ by (3)

Let ψ be the angle through which the forces are to be turned to get a couple of moment G again.

Then $G = P \,.\, AB \sin (\theta + \psi)$

$= G \cos \psi + H \sin \psi$

or, $G(1 - \cos \psi) = H \sin \psi$

or, $\dfrac{1 - \cos \psi}{\sin \psi} = \dfrac{H}{G}$

or, $\dfrac{H}{G} = \dfrac{2 \sin^2 \frac{\psi}{2}}{2 \sin \frac{\psi}{2} \cos \frac{\psi}{2}} = \tan \dfrac{\psi}{2}$

$\therefore$ $\psi = 2 \tan^{-1} (H/G)$ is the required angle.

Exercise 19.1

1. Forces of magnitudes 1, 2, 3, 4, $2\sqrt{2}$ act respectively along the sides AB, BC, CD, DA and the diagonal AC of the square $ABCD$. Show that their resultant is a couple and find its moment.
2. Forces equal to 3, 5, 3 and 5 lbs.wt. act respectively along the sides of a square taken in order. Find their resultant.
3. Show that the forces 3, 8, 7, 11 and 5 lbs.wt. acting respectively along the sides AB, BC, CD, DA and the diagonal AC of a rectangle $ABCD$ are equivalent to a couple, if $AB = 6$ ft. and $BC = 4\frac{1}{2}$ ft. and show that the moment of the couple is $79\frac{1}{2}$ ft.lbs.
4. P and Q are like parallel forces. If a couple, each of whose forces is F, and whose arm is a in the plane of P and Q, is combined

with them, show that the resultant is displaced through a distance $\dfrac{Fa}{P+Q}$.

5. A horizontal bar *AB*, without weight, is acted on by a vertically downward force of 1 lb.wt. at *A*, a vertically upward force of 1 lb.wt. at *B*, and a downward force of 5 lbs.wt. at a given point *C* inclined to the bar at an angle of 30°. Find at what point of the bar must a force be applied to balance these, and find also its magnitude and direction.

6. *D, E, F* divide the sides *BC, CA, AB* respectively of an equilateral triangle *ABC* of side a in the ratio 5 : 1. Three forces each equal to *P* act at *D, E, F* perpendicular to the sides and outwards from the triangle. Show that they are equivalent to a couple of moment Pa.

7. Three forces, proportional to the sides of a triangle, act perpendicularly to these sides. Prove that they are either in equilibrium or form a couple.

8. Three parallel forces *P, Q, R* acting at the angular points of the triangle *ABC*, are in equilibrium when they act at right angles to *BC*. Prove that if each line of action be turned through an angle α in the same direction, they will be equivalent to a couple of moment $Pb \sin \alpha \sin C$.

9. If the algebraic sum of the moments of two given coplanar forces about any point in their plane be a non-zero constant, show that the given forces form a couple.

10. *ABCD* is a rectangle such that $AB = CD = a$ and $BC = DA = b$. Forces *P, P* act along *AD* and *CB*; forces *Q, Q* act along *AB* and *CD*. Prove that the perpendicular distance between the resultant of the forces *P, Q* at *A* and the resultant of the forces *P, Q* at *C* is $\dfrac{Pa - Qb}{\sqrt{P^2 + Q^2}}$.

11. Forces *P* and *Q* act at *A*, and are completely represented by *AB* and *AC*, sides of a triangle *ABC*. Find a third force *R* such that the three forces together may be equivalent to a couple whose moment is represented by half the area of a triangle.

12. Four forces are completely represented by the sides *AB, BC, CD, DA* of a quadrilateral *ABCD*. Show that they are equivalent to a couple, consisting of two equal forces through *B* and *D*.

20 Centre of Gravity

In dealing with the problems of mechanics, we frequently assume that the weight of a straight uniform heavy rod acts at its middle point, and generally, that the weight of a rigid body may be supposed to be concentrated at a single point, called the **centre of gravity** of the body. Before going to the actual definition of the centre of gravity, it will be helpful to develop the idea of the **centre of parallel forces.**

20.1 Centre of parallel forces

We shall now find out the resultant of any number of parallel forces of given magnitudes applied at given points of a rigid body, not necessarily in the same plane.

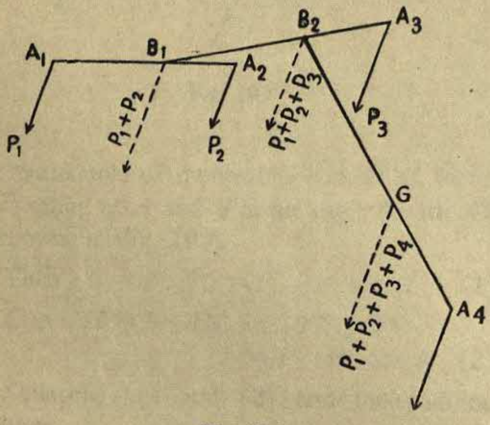


Fig. 20.1

Let P_1, P_2, P_3 .... be a system of parallel forces acting at A_1, A_2, A_3, ... of a rigid body. $A_1 A_2$ is joined. The forces P_1 and P_2 are equivalent to a single resultant force $(P_1 + P_2)$, parallel to them, acting at B_1 on $A_1 A_2$ such that $A_1 B_1 : A_2 B_1 = P_2 : P_1$.

Let us now join $B_1 A_3$ and compound the force $(P_1 + P_2)$ at B_1 with the force P_3 at A_3. The force $(P_1 + P_2)$ at B_1 and P_3 at A_3 are equivalent to a single resultant force, parallel to them, $(P_1 + P_2 + P_3)$ acting at B_2 on $B_1 A_3$ such that $B_1B_2 : A_3B_2 = P_3 : (P_1 + P_2)$.

Let us now join B_2A_4. The force $(P_1 + P_2 + P_3)$ at B_2 and P_4 at A_4 are equivalent to a single resultant $(P_1 + P_2 + P_3 + P_4)$, parallel to them, acting at G on B_2A_4 such that

$$B_2G : A_4G = P_4 : (P_1 + P_2 + P_3)$$

This resultant is therefore the resultant of the four forces P_1, P_2, P_3, P_4 acting at A_1, A_2, A_3, A_4 respectively.

Proceeding in this way, we may find the

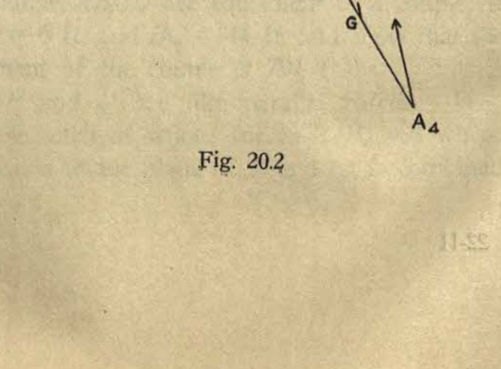


Fig. 20.2

resultant of any number of parallel forces acting at different points of the body.

If the forces remain unchanged in magnitude but act at the same points in some direction (remaining parallel to themselves) other than that as shown in Fig. 20.1, their resultant will still pass through the same point G as before [see Fig. 20.2].

The point G is called the **centre of parallel forces** P_1, P_2, P_3, P_4.

We may, therefore, conclude :

The resultant of any number of parallel forces passes through a certain point whose position depends only on the magnitudes and the points of application of the forces and not on their direction. This point is called the centre of the parallel forces.

20.2 Centre of gravity

A rigid body consists of an infinite number of particles each of which has its own weight acting vertically downwards. Thus the weights of the particles form a system of like parallel forces. This system of like parallel forces may be supposed to be replaced by its resultant force, and it is convenient to speak of this resultant as the **weight of the body.** The point of application of this resultant is called the **centre of gravity,** or briefly **C. G.,** of the body.

The particles of the body will have their positions and weights unchanged unless the shape or size of the body changes. We may therefore say that the C. G. of a rigid body is fixed with regard to itself or that it moves with the body.

We may, therefore, define the centre of gravity as follows :

The centre of gravity of a body is that point, fixed relative to the body, at which the whole weight of the body may always be supposed to act.

REMARKS

1. The centre of gravity may not always lie in the body itself ; for example, the centre of gravity of a circular ring of wire is at the centre of the circle and not in the material of the wire itself.

2. *A body cannot have more than one C. G.*: If possible, let a body have two centres of gravity at G and H. Then the line of action of the resultant weight of the body must have to pass through both G and H, and would therefore be parallel to the line GH. This is possible only when the line GH is vertical. Hence the body cannot have two centres of gravity.

20.3 Some definitions

(i) A **lamina** is a sheet of material whose thickness is so small that it may be regarded as a distribution of matter over an area. A **uniform lamina** is one which is of the same thickness and formed of the same substance throughout.

(ii) **The C.G. of an area or surface** means C.G. of uniform lamina covering that area or surface.

(iii) **The C.G. of a line** (either straight or curved) means the C.G. of a thin uniform wire placed along that line.

(iv) A wire, stick, rod or beam is said to be **uniform** when it is of the same cross section and of the same material throughout its length.

20.4 Centre of mass or C. M.

If the body be removed from the influence of gravity by being carried right away from the earth into outer space or if it be imagined to be taken to the centre of the earth, the gravity will not exist and, therefore, the body will have no C.G. But the body will still have its **mass** and the analogy suggests that it will have a **centre of mass.**

All the particles of the body are acted on by a system of parallel forces, the force on each particle being proportional to its mass. The centre of this system of parallel forces is called the **centre of mass** (briefly denoted by C.M.) of the body.

Since the weight of each particle of the body is proportional to its mass, we may call the C.G. of the body as its C.M., but not vice versa because existence of the C.G. is independent on the existence of gravity.

20.5 Determination of centre of gravity of a system of rectilinear particles

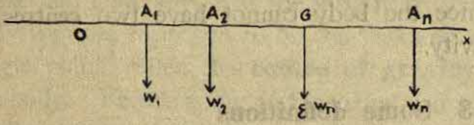


Fig. 20.3

Let a system of particles having weights $w_1, w_2, \ldots\ldots, w_n$ lie at $A_1, A_2, \ldots, A_n$ respectively on the straight line OX. Let $x_1, x_2, \ldots, x_n$ be the distances of the points $A_1, A_2, \ldots, A_n$ respectively from O. Let the resultant weight $(w_1 + w_2 + \cdots + w_n) = \sum_{k=1}^{n} w_k$ act at G where $OG = \bar{x}$. Then G is the C. G. of the system of particles.

Taking moments about O, we have

$$w_1x_1 + w_2x_2 + \cdots + w_nx_n = (w_1 + w_2 + \cdots + w_n)\bar{x}$$

$$\text{or,}\quad \sum_{k=1}^{n} w_kx_k = \bar{x}\sum_{k=1}^{n} w_k$$

$$\therefore\quad \bar{x} = \frac{\sum_{1}^{n} w_kx_k}{\sum_{1}^{n} w_k} \tag{1}$$

If $m_1, m_2, \ldots, m_n$ be the masses of the particles at $A_1, A_2, \ldots, A_n$ respectively, we have

$$w_k = gm_k,\ k = 1, 2, \ldots, n.$$

$$\therefore\quad \bar{x} = \frac{\Sigma\, gm_kx_k}{\Sigma\, gm_k} = \frac{\Sigma\, m_kx_k}{\Sigma\, m_k}. \tag{2}$$

20.6 Determination of centre of gravity of a system of coplanar particles

Let OX, OY be any two mutually perpendicular lines on the plane on which the particles lie.

Let the particles of weights $w_1, w_2, \ldots, w_n$ be situated at $A_1, A_2, \ldots, A_n$ whose coordinates are

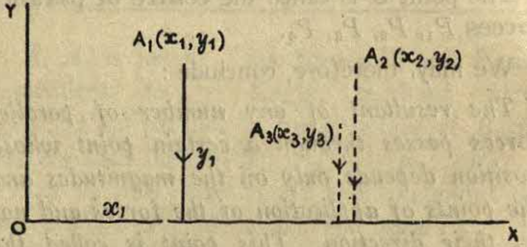


Fig. 20.4

(x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$ respectively referred to OX, OY as the rectangular axes. Let $G(\bar{x}, \bar{y})$ be the centre of gravity of the system.

We know that the C.G. is the centre of a system of parallel forces and is independent of the direction in which the forces act. Thus, if the lines of action of the weights are all rotated to make them parallel to y-axis, the C.G. still remains at G, i.e. the resultant weight $(w_1 + w_2 + \cdots + w_n)$ acts at G in a direction parallel to y-axis.

∴ Taking moments about O, we get

$$w_1x_1 + w_2x_2 + \cdots + w_nx_n = (w_1 + w_2 + \cdots + w_n)\bar{x}$$

$$\text{or,}\quad \bar{x} = \frac{\sum_{k=1}^{n} w_kx_k}{\sum_{k=1}^{n} w_k} \tag{1}$$

Similarly, if the lines of action of all the weights are rotated so as to make them parallel to the x-axis, we have by taking moments about O,

$$\bar{y} = \frac{\sum_{k=1}^{n} w_k y_k}{\sum_{k=1}^{n} w_k}. \qquad (2)$$

If the masses of the particles be $m_1, m_2, \ldots m_n$ respectively, we have

$$\bar{x} = \frac{\sum_1^n m_k x_k}{\sum_1^n m_k}, \quad \bar{y} = \frac{\sum_1^n m_k y_k}{\sum_1^n m_k} \qquad (3)$$

20.7 Centre of gravity of a plane area

If in place of discrete particles, we take into consideration a continuous distribution of matter, then the summation in the formulae for $\bar{x}$ and $\bar{y}$ can be replaced by integration. Thus the position of the centre of gravity $G(\bar{x}, \bar{y})$ of a body is given by

$$\bar{x} = \frac{\int x\, dw}{\int dw}, \quad \bar{y} = \frac{\int y\, dw}{\int dw} \qquad (1)$$

where (x, y) stands for the C.G. of an elementary part of weight dw of the body.

If dm be the mass of the elementary part of weight dw, we have $dw = g\, dm$ and therefore,

$$\bar{x} = \frac{\int x\, dm}{\int dm}, \quad \bar{y} = \frac{\int y\, dm}{\int dm} \qquad (2)$$

If ρ be the density per unit area and δS denotes an elementary area, we have $\delta m = \rho \delta S$ and then formulae (2) may be written as

$$\bar{x} = \frac{\int x\rho\, dS}{\int \rho\, dS}, \quad \bar{y} = \frac{\int y\rho\, dS}{\int \rho\, dS} \qquad (3)$$

When the density is uniform throughout the plane area, ρ becomes a constant and therefore, (3) reduces to

$$\bar{x} = \frac{\int x\, dS}{\int dS}, \quad \bar{y} = \frac{\int y\, dS}{\int dS} \qquad (4)$$

We shall now consider the following cases :

CASE I : *C.G. of a uniform plane area bounded by the curve $y = f(x)$, the axis of x and the ordinates at $x = a$, $x = b$.*

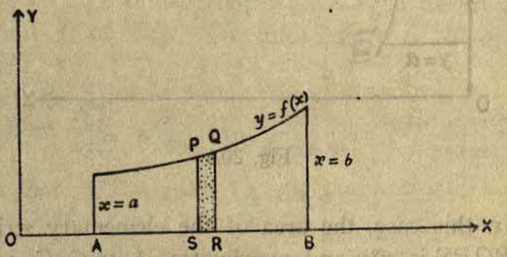


Fig. 20.5

Let the coordinates of two neighbouring points P and Q on the curve $y = f(x)$ be (x, y) and $(x + \delta x, y + \delta y)$ respectively. The whole area is divided into thin elementary strips parallel to the y-axis. Let $PQRS$ be such a strip.

The area of this elementary strip is $y\delta x$ and the coordinates of its centre of gravity are $\left(x, \frac{y}{2}\right)$ approximately.

Using these results in (4), we have

$$\bar{x} = \frac{\int_a^b xy\, dx}{\int_a^b y\, dx}, \quad \bar{y} = \frac{1}{2}\frac{\int_a^b y^2\, dx}{\int_a^b y\, dx} \qquad (5)$$

REMARKS

When the plane area is symmetrical about the x-axis, area of the elementary strip will be $2y\delta x$ and its C.G. will be (x, o).

$$\text{Hence } \bar{x} = \frac{\int_a^b xy\, dx}{\int_a^b y\, dx}, \quad \bar{y} = 0. \qquad (6)$$

CASE II : *C. G. of a uniform plane area bounded by the curve $x = f(y)$, the axis of y and the abscissae at $y = a$ and $y = b$.*

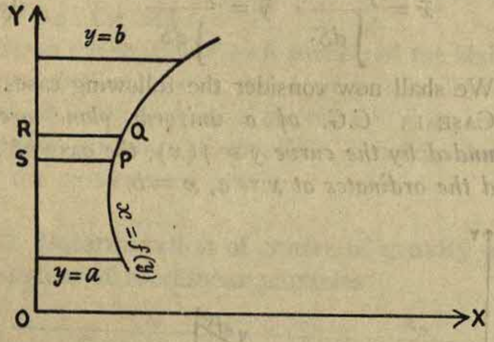


Fig. 20.6

In this case, the area of the elementary strip *PQRS* is $x\delta y$ and coordinates of its C. G. are $\left(\frac{x}{2}, y\right)$ approximately. Using these results in (4), we have

$$\bar{x} = \frac{1}{2}\frac{\int_a^b x^2\,dy}{\int_a^b x\,dy}, \quad \bar{y} = \frac{\int_a^b xy\,dy}{\int_a^b x\,dy}. \qquad (7)$$

REMARKS

When the plane area is symmetrical about the y-axis, area of the elementary strip will be $2x\delta y$ and its C.G. will be (o, y).

Hence from (4), $\bar{x} = 0$, $\bar{y} = \dfrac{\int_a^b xy\,dy}{\int_a^b x\,dy}$.

20.8 Determination of C.G. of some bodies

20.8(*a*) *C.G. of a thin uniform rod*

Let *AB* be a thin uniform straight rod of length $2a$. Let us take *A* as the origin and *AB* as the axis of x. Let *P*, *Q* be two neighbouring points on the rod where $AP = x$, $AQ = x + dx$.

If ρ be the density and α the area of cross-section of the rod, element of mass at *P* is $dm = \rho\alpha dx$.

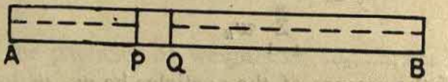


Fig. 20.7

The rod being uniform, α and ρ are constants.

$$\therefore \quad \bar{x} = \frac{\int_0^{2a} x\,.\,\rho\alpha dx}{\int_0^{2a} \rho\alpha\,dx} = \frac{\int_0^{2a} x\,dx}{\int_0^{2a} dx}$$

$$= \frac{\frac{1}{2}\left|x^2\right|_0^{2a}}{\left|x\right|_0^{2a}} = \frac{\frac{1}{2} \times 4a^2}{2a} = a.$$

Hence the C.G. of a thin uniform rod is at its middle point.

20.8(*b*) *C.G. of a uniform rectangular lamina*

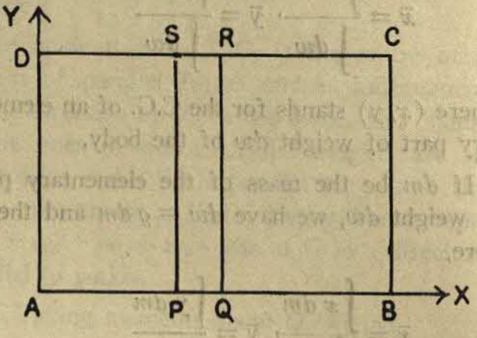


Fig. 20.8

Let *ABCD* be a uniform rectangular lamina of sides $AB = 2a$, $BC = 2b$. We take the axes of x and y along *AB* and *AD* respectively, *A* being the origin.

The lamina is divided into thin elementary strips parallel to *AD*. Let us take such an elementary strip *PQRS* of thickness dx at $P(x, y)$.

Mass of this elementary strip $= P\,.\,2bdx$ where ρ is the uniform surface-density of the lamina.

Coordinates of the C.G. of the strip are (x, b). If $(\bar{x}, \bar{y})$ be the coordinates of the C.G. of the lamina, we have

$$\bar{x} = \frac{\int_0^{2a} x.2b\rho\,dx}{\int_0^{2a} 2b\rho\,dx} = \frac{\int_0^{2a} x\,dx}{\int_0^{2a} dx} = \frac{\left|\frac{x^2}{2}\right|_0^{2a}}{\left|x\right|_0^{2a}}$$

$$= \frac{2a^2}{2a} = a,$$

$$\text{and } \bar{y} = \frac{\int_0^{2a} b.2b\rho\,dx}{\int_0^{2a} 2b\rho\,dx} = \frac{b\int_0^{2a} dx}{\int_0^{2a} dx} = b.$$

$\therefore$ *The C.G. of the rectangular lamina is* at the point (a, b), i.e. *at the midpoint of the line segment joining the midpoints of two opposite sides.*

20.8 (*c*) *C.G. of a uniform triangular lamina*

Let ABC be a uniform triangular lamina. The lamina is divided into thin elementary strips parallel to the base BC. AM is drawn perpendicular to BC.

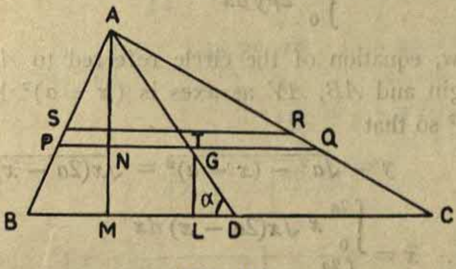


Fig. 20.9

We take M as the origin, MC as the x-axis and MA as the y-axis.

Let $PQRS$ be an elementary strip of thickness dy at $N(o, y)$. Let D be the midpoint of BC and $\angle ADB = \alpha$.

Taking $AD = h$, $AM = h \sin \alpha$,

$$AN = h \sin \alpha - y.$$

$$\text{Now, } \frac{PQ}{BC} = \frac{AN}{AM}; \text{ or, } \frac{PQ}{a} = \frac{h \sin \alpha - y}{h \sin \alpha}$$

so that $PQ = \dfrac{a(h \sin \alpha - y)}{h \sin \alpha}$.

$\therefore$ Mass of the elementary strip $PQRS$

$$= \frac{\rho a (h \sin \alpha - y)}{h \sin \alpha}\,dy.$$

The C.G. of the strip $PQRS$ is at the midpoint T of PQ. The coordinates of T are (NT, y).

$$\text{Now } \frac{NT}{MD} = \frac{AN}{AM}; \text{ or, } \frac{NT}{h \cos \alpha} = \frac{h \sin \alpha - y}{h \sin \alpha}$$

so that $NT = \cot \alpha\ (h \sin \alpha - y)$.

$\therefore$ The C.G. of the strip $PQRS$ is at $\{(h \sin \alpha - y) \cot \alpha, y\}$. Let $(\bar{x}, \bar{y})$ be the coordinates of the C.G. of the lamina.

Then $\bar{x} =$

$$\frac{\int_0^{h \sin \alpha} (h \sin \alpha - y) \cot \alpha \times \frac{\rho a (h \sin \alpha - y)}{h \sin \alpha}\,dy}{\int_0^{h \sin \alpha} \frac{\rho a (h \sin \alpha - y)}{h \sin \alpha}\,dy}$$

$$= \frac{\cot \alpha \int_0^{h \sin \alpha} (h \sin \alpha - y)^2 dy}{\int_0^{h \sin \alpha} (h \sin \alpha - y)\,dy}$$

$$= \frac{\cot \alpha \left[h^2 \sin^2\alpha \int_0^{h \sin \alpha} dy - 2h \sin \alpha \int_0^{h \sin \alpha} y dy + \int_0^{h \sin \alpha} y^2 dy\right]}{h \sin \alpha \int_0^{h \sin \alpha} dy - \int_0^{h \sin \alpha} y\,dy}$$

$$= \frac{\cot \alpha \left[h^2 \sin^2\alpha \cdot h \sin \alpha - 2h \sin \alpha \cdot \frac{h^2 \sin^2\alpha}{2} + \frac{h^3 \sin^3\alpha}{3}\right]}{h \sin \alpha \cdot h \sin \alpha - \frac{h^2 \sin^2\alpha}{2}}$$

$$= \frac{\cot\alpha\cdot\dfrac{h^3\sin^3\alpha}{3}}{\dfrac{h^2\sin^2\alpha}{2}} = \frac{2}{3}h\cos\alpha$$

and $$\bar{y} = \frac{\displaystyle\int_0^{h\sin\alpha} y\times\frac{\rho a(h\sin\alpha - y)}{h\sin\alpha}\,dy}{\displaystyle\int_0^{h\sin\alpha}\frac{\rho a(h\sin\alpha - y)}{h\sin\alpha}\,dy}$$

$$= \frac{h\sin\alpha\displaystyle\int_0^{h\sin\alpha} y\,dy - \int_0^{h\sin\alpha} y^2\,dy}{h\sin\alpha\displaystyle\int_0^{h\sin\alpha} dy - \int_0^{h\sin\alpha} y\,dy}$$

$$= \frac{h\sin\alpha\cdot\dfrac{h^2\sin^2\alpha}{2} - \dfrac{h^3\sin^3\alpha}{3}}{h\sin\alpha\cdot h\sin\alpha - \dfrac{h^2\sin^2\alpha}{2}}$$

$$= \frac{\dfrac{h^3\sin^3\alpha}{6}}{\dfrac{h^2\sin^2\alpha}{2}} = \frac{1}{3}h\sin\alpha.$$

Hence the C.G. of the triangular lamina is at the point $(\frac{2}{3}h\cos\alpha, \frac{1}{3}h\sin\alpha)$. We shall now show that this point is actually the centroid of $\triangle ABC$.

If G be the centroid of $\triangle ABC$ and GL be perpendicular to BC, the coordinates of G are (ML, GL).

$$\begin{aligned} \text{Now } ML &= MD - LD = h\cos\alpha \\ &\quad - GD\cos\alpha = h\cos\alpha - \tfrac{1}{3}h\cos\alpha \\ &= \tfrac{2}{3}h\cos\alpha, \\ GL &= GD\sin\alpha = \tfrac{1}{3}h\sin\alpha. \\ &\qquad [\because GD = \tfrac{1}{3}AD = \tfrac{1}{3}h] \end{aligned}$$

Hence *the C.G. of the triangular lamina is at its centroid.*

20.8 (*d*) *C.G. of a uniform circular lamina*

Let a be the radius and O the centre of the uniform circular lamina of surface-density (say) ρ. A diameter AB is taken as the x-axis and the tangent at A is taken as the y-axis, A being the origin.

The lamina is divided into thin elementary strips parallel to the y-axis. Let us consider such an elementary strip $PQRS$ of thickness dx at the point (x, o). Mass of this elementary strip $= \rho.2y.dx = 2\rho y dx$ where (x, y) are the coordinates of P.

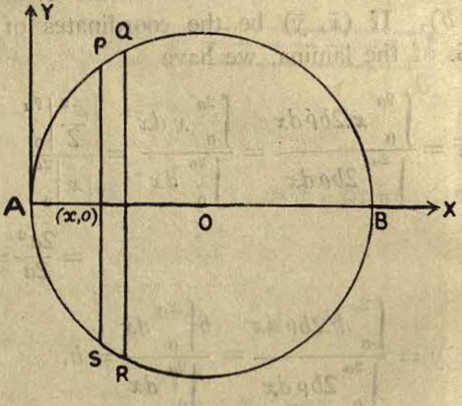


Fig. 20.10

Coordinates of the C.G. of the strip are (x, o) [by case (i) above].

Let $(\bar{x}, \bar{y})$ be the coordinates of the C.G. of the circular lamina.

$$\text{Then } \bar{x} = \frac{\displaystyle\int_0^{2a} x.2\rho y\,dx}{\displaystyle\int_0^{2a} 2\rho y\,dx},\quad \bar{y} = 0.$$

Now, equation of the circle referred to A as origin and AB, AY as axes is $(x-a)^2 + y^2 = a^2$ so that

$$y = \sqrt{a^2 - (x-a)^2} = \sqrt{x(2a - x)}.$$

$$\therefore\quad \bar{x} = \frac{\displaystyle\int_0^{2a} x\sqrt{x(2a-x)}\,dx}{\displaystyle\int_0^{2a}\sqrt{x(2a-x)}\,dx}$$

Let us put $x = 2a\sin^2\theta$ so that $dx = 4a\sin\theta\cos\theta d\theta$.

Also,

$$\theta = 0 \text{ when } x = 0, \text{ and } \theta = \frac{\pi}{2} \text{ when } x = 2a.$$

$$\therefore\ \bar{x} = \frac{\displaystyle\int_0^{\pi/2} (2a)^{\frac{3}{2}} \sin^3\theta \times \sqrt{2a}\cos\theta \times 4a\sin\theta\cos\theta d\theta}{\displaystyle\int_0^{\pi/2} \sqrt{2a}\sin\theta \times \sqrt{2a}\cos\theta \times 4a\sin\theta\cos\theta\, d\theta}$$

$$= \frac{2a\displaystyle\int_0^{\pi/2} \sin^4\theta\cos^2\theta\, d\theta}{\displaystyle\int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta}$$

$$= \frac{\dfrac{a}{2}\displaystyle\int_0^{\pi/2} \sin^2\theta\sin^2 2\theta\, d\theta}{\dfrac{1}{4}\displaystyle\int_0^{\pi/2} \sin^2 2\theta\, d\theta}$$

$$= \frac{2a\displaystyle\int_0^{\pi/2} \sin^2\theta\sin^2 2\theta\, d\theta}{\displaystyle\int_0^{\pi/2} \sin^2 2\theta\, d\theta}$$

$$= \frac{\dfrac{a}{2}\displaystyle\int_0^{\pi/2} (1-\cos 2\theta)(1-\cos 4\theta)\, d\theta}{\dfrac{1}{2}\displaystyle\int_0^{\pi/2} (1-\cos 4\theta)\, d\theta}$$

$$= \frac{a\left[\displaystyle\int_0^{\pi/2} d\theta - \int_0^{\pi/2}\cos 2\theta - \int_0^{\pi/2}\cos 4\theta d\theta + \int_0^{\pi/2}\cos 2\theta\cos 4\theta\, d\theta\right]}{\displaystyle\int_0^{\pi/2} d\theta - \int_0^{\pi/2}\cos 4\theta\, d\theta}$$

$$= \frac{a\left[\Big|\theta\Big|_0^{\pi/2} - \Big|\dfrac{\sin 2\theta}{2}\Big|_0^{\pi/2} - \Big|\dfrac{\sin 4\theta}{4}\Big|_0^{\pi/2} + \dfrac{1}{2}\Big|\dfrac{\sin 6\theta}{6} + \dfrac{\sin 2\theta}{2}\Big|_0^{\pi/2}\right]}{\Big|\theta\Big|_0^{\pi/2} - \Big|\dfrac{\sin 4\theta}{4}\Big|_0^{\pi/2}}$$

$$= \frac{a\left[\dfrac{\pi}{2} - 0 - 0 + 0\right]}{\dfrac{\pi}{2} - 0} = a.$$

Hence *the C.G. of the circular lamina is at* (a, o) *which is actually the centre of the lamina.*

REMARKS

1. The lamina may also be divided into thin strips parallel to the diameter AB.

2. The centre O may also be taken as the origin. The equation of the circle in that case becomes $x^2 + y^2 = a^2$ and the limits of integration are $x = -a$ and $x = +a$.

20.9 Illustrative examples

EXAMPLE 1

AB and AC are two uniform rods of lengths $2a$ and $2b$ respectively. If $\angle BAC = \theta$, prove that the distance from A of the C.G. of the two rods is

$$\left(\frac{a^4 + 2a^2b^2\cos\theta + b^4}{a+b}\right)^{\frac{1}{2}}$$

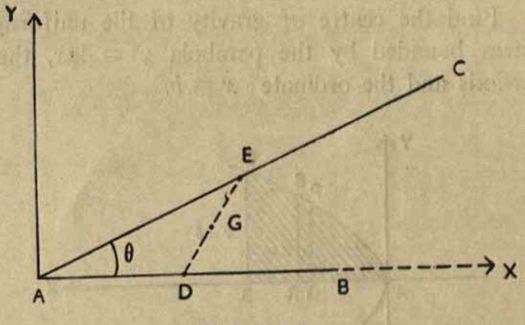


Fig. 20.11

Let D, E be respectively the midpoints of AB and AC.

Then the weight $2aw$ of the rod AB acts at D and the weight $2bw$ of the rod AC acts at E where w is the weight per unit length of the rods.

Hence the centre of gravity G of the system must lie at some point on DE.

To determine the distance AG, let us take the x-axis along the line AB and the y-axis along a direction AY perpendicular to AB.

Referred to this coordinate system, the coordinates of D, E are respectively (a, o) and $(b\cos\theta,\ b\sin\theta)$.

If the coordinates of G be $(\bar{x},\ \bar{y})$, we have

$$\bar{x} = \frac{x_1W_1 + x_2W_2}{W_1 + W_2}$$

$$= \frac{a.2aw + b\cos\theta.2bw}{2aw + 2bw}$$

$$= \frac{a^2 + b^2\cos\theta}{a+b},$$

$$\bar{y} = \frac{a.0 + b\sin\theta.2bw}{2aw + 2bw}$$

$$= \frac{b^2\sin\theta}{a+b}.$$

$$\therefore\quad AG = (\bar{x}^2 + \bar{y}^2)^{\frac{1}{2}}$$

$$= \frac{(a^4 + 2a^2b^2\cos\theta + b^4)^{\frac{1}{2}}}{a+b}.$$

EXAMPLE 2

Find the centre of gravity of the uniform area bounded by the parabola $y^2 = 4ax$, the x-axis and the ordinate $x = b$.

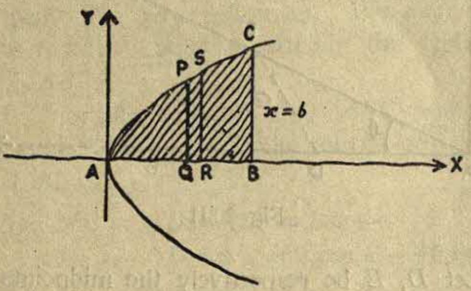


Fig. 20.12

Let A be the vertex of the parabola

$$y^2 = 4ax \tag{1}$$

and BC the ordinate $x = b$.

We are to find out the C.G. of the area ABC. Let this area be divided into thin elementary strips parallel to the y-axis AY. Let us consider such an elementary strip $PQRS$ of breadth dx at $P(x, y)$.

Mass of this elementary strip $= \rho\,.\,y\,.\,dx = \rho y dx$ where ρ is the uniform surface-density of the area ABC.

Coordinates of C. G. of this strip are $\left(x, \frac{y}{2}\right)$.

Let $(\bar{x},\ \bar{y})$ be the coordinates of the C.G. of the area ABC.

$$\text{Then } \bar{x} = \frac{\int_0^b x.\rho y\,dx}{\int_0^b \rho y\,dx} = \frac{\int_0^b x.2\sqrt{ax}\,dx}{\int_0^b 2\sqrt{ax}\,dx}, \quad \text{by (1)}$$

$$= \frac{\int_0^b x^{\frac{3}{2}}\,dx}{\int_0^b x^{\frac{1}{2}}\,dx} = \frac{\frac{2}{5}b^{\frac{5}{2}}}{\frac{2}{3}b^{\frac{3}{2}}} = \frac{3}{5}b$$

$$\text{and}\quad \bar{y} = \frac{\int_0^b \frac{1}{2}y.\rho y\,dx}{\int_0^b \rho y\,dx} = \frac{1}{2}\frac{\int_0^b y^2\,dx}{\int_0^b y\,dx}$$

$$= \frac{1}{2}\frac{\int_0^b 4ax\,dx}{\int_0^b 2\sqrt{ax}\,dx}$$

$$= \sqrt{a}\,\frac{\int_0^b x\,dx}{\int_0^b x^{\frac{1}{2}}\,dx} = \sqrt{a}\cdot\frac{\frac{b^2}{2}}{\frac{2}{3}b^{\frac{3}{2}}}$$

$$= \frac{3}{4}\sqrt{ab}.$$

EXAMPLE 3

Find the centre of gravity of the segment of a uniform circular disc.

Let the equation of the uniform circular disc be

$$x^2 + y^2 = a^2 \tag{1}$$

referred to its centre o as origin, a being its radius.

Let us consider the segment $ABDCA$ of the disc subtending an angle 2α at the centre O.

The segment is divided into thin elementary strips parallel to BC. Let us consider such an elementary strip $PQRS$ of thickness dx at $P(x, y)$.

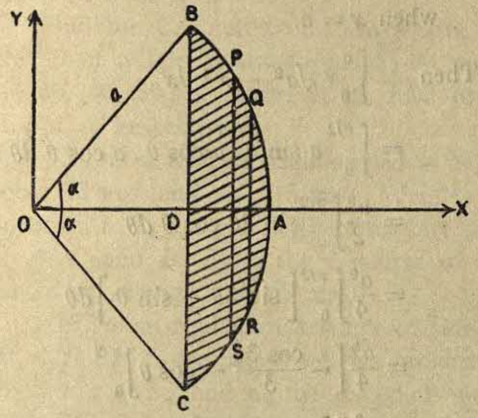


Fig. 20.13

Mass of the strip $PQRS = \rho \,.\, 2ydx = 2\rho ydx$ where ρ is the uniform surface density of the segment.

Coordinates of the C.G. of the strip $PQRS$ are (x, o).

Let $(\bar{x}, \bar{y})$ be the coordinates of the C.G. of the segment $ABDCA$.

$$\text{Then} \quad \bar{x} = \frac{\displaystyle\int_{OD}^{OA} x \,.\, 2\rho y \, dx}{\displaystyle\int_{OD}^{OA} 2\rho y \, dx}$$

$$= \frac{\displaystyle\int_{a\cos\alpha}^{a} x \sqrt{a^2 - x^2} \, dx}{\displaystyle\int_{a\cos\alpha}^{a} \sqrt{a^2 - x^2} \, dx}, \text{ by (1)}$$

$$\bar{y} = 0.$$

Let us put $x = a \cos\theta$ so that $dx = -a \sin\theta \, d\theta$. Also $\theta = \alpha$ when $x = a\cos\alpha$, and $\theta = o$ when $x = a$.

$$\therefore \int_{a\cos\alpha}^{a} x\sqrt{a^2 - x^2} \, dx$$

$$= -\int_{\alpha}^{0} a\cos\theta \,.\, a\sin\theta \,.\, a\sin\theta \, d\theta$$

$$= +\frac{a^3}{2}\int_{0}^{\alpha} \sin\theta \sin 2\theta \, d\theta$$

$$= \frac{a^3}{4}\int_{0}^{\alpha}\Big[\cos\theta - \cos 3\theta\Big] d\theta$$

$$= \frac{a^3}{4}\Big[\sin\theta - \frac{\sin 3\theta}{3}\Big]_{\theta=0}^{\theta=\alpha}$$

$$= \frac{a^3}{4}\Big[\sin\alpha - \frac{\sin 3\alpha}{3}\Big]$$

$$= \frac{a^3}{12}\Big[3\sin\alpha - \sin 3\alpha\Big]$$

$$= \frac{a^3}{12}\Big[3\sin\alpha - (3\sin\alpha - 4\sin^3\alpha)\Big]$$

$$= \frac{a^3}{3}\sin^3\alpha,$$

$$\text{and} \int_{a\cos\alpha}^{a} \sqrt{a^2 - x^2} \, dx$$

$$= -\int_{\alpha}^{0} a\sin\theta \,.\, a\sin\theta \, d\theta$$

$$= \frac{a^2}{2}\int_{0}^{\alpha}(1 - \cos 2\theta) \, d\theta$$

$$= \frac{a^2}{2}\Big|\theta - \frac{\sin 2\theta}{2}\Big|_{0}^{\alpha}$$

$$= \frac{a^2}{2}\Big[\alpha - \sin\alpha\cos\alpha\Big].$$

$$\therefore \quad \bar{x} = \frac{\frac{a^3}{3}\sin^3\alpha}{\frac{a^2}{2}(\alpha - \sin\alpha\cos\alpha)}$$

$$= \frac{2}{3}a \cdot \frac{\sin^3\alpha}{\alpha - \sin\alpha\cos\alpha},$$

$$\bar{y} = 0$$

give the position of the C.G. of the segment.

EXAMPLE 4

Find the C.G. of a quadrant of a uniform elliptic lamina.

Let the quadrant OAB of the uniform elliptic lamina be divided into thin elementary strips parallel to the y-axis. Let us take such an elementary strip *PQRS* of breadth dx at $P(x, y)$.

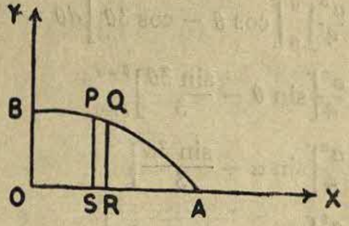


Fig. 20.14

Let the equation of the rim of the lamina be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ so that}$$

$$y=\frac{b}{a}\sqrt{a^2-x^2} \qquad (1)$$

Mass of the strip *PQRS* is $dm = \rho \,.\, y \,.\, dx = \rho y dx$ where ρ is the uniform surface-density of the lamina.

Coordinates of C.G. of the strip *PQRS* are $(x, \frac{1}{2}y)$.

Let $(\bar{x}, \bar{y})$ be the coordinates of C.G. of the quadrant *OAB*.

Then

$$\bar{x}=\frac{\int_0^a x \,.\, \rho y\, dx}{\int_0^a \rho y\, dx}=\frac{\int_0^a xy\, dx}{\int_0^a y\, dx}$$

$$=\frac{\frac{b}{a}\int_0^a x\sqrt{a^2-x^2}\,dx}{\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx} \text{ by (1),}$$

$$\text{and } \bar{y}=\frac{\int_0^a \frac{1}{2}y \,.\, \rho y\, dx}{\int_0^a \rho y\, dx}=\frac{1}{2}\cdot\frac{\int_0^a y^2\, dx}{\int_0^a y\, dx}$$

$$=\frac{1}{2}\cdot\frac{\frac{b^2}{a^2}\int_0^a (a^2-x^2)\,dx}{\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx}$$

$$=\frac{b}{2a}\cdot\frac{\int_0^a (a^2-x^2)\,dx}{\int_0^a \sqrt{a^2-x^2}\,dx} \text{ by (1).}$$

Let $x = a \sin\theta$ so that $dx = a\cos\theta\, d\theta$.

Also, $\theta = 0$ when $x = 0$, and $\theta=\frac{\pi}{2}$ when $x = a$.

$$\text{Then } \int_0^a x\sqrt{a^2-x^2}\,dx$$

$$=\int_0^{\pi/2} a\sin\theta \,.\, a\cos\theta \,.\, a\cos\theta\, d\theta$$

$$=\frac{a^3}{2}\int_0^{\pi/2}\sin 2\theta\cos\theta\, d\theta$$

$$=\frac{a^3}{4}\int_0^{\pi/2}\Big[\sin 3\theta+\sin\theta\Big]d\theta$$

$$=\frac{a^3}{4}\Big[-\frac{\cos 3\theta}{3}-\cos\theta\Big]_0^{\pi/2}$$

$$=\frac{a^3}{4}\Big[\frac{1}{3}+1\Big]=\frac{a^3}{3};$$

$$\int_0^a \sqrt{a^2-x^2}\,dx=a^2\int_0^{\pi/2}\cos^2\theta\, d\theta$$

$$=\frac{a^2}{2}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta=\frac{a^2}{2}\Big|\theta+\frac{\sin 2\theta}{2}\Big|_0^{\pi/2}$$

$$=\frac{a^2}{2}\;\frac{\pi}{2}=\frac{\pi a^2}{4};$$

$$\int_0^a (a^2-x^2)\,dx=a^2\Big|x\Big|_0^a-\frac{1}{3}\Big|x^3\Big|_0^a$$

$$=a^3-\frac{a^3}{3}=\frac{2}{3}a^3.$$

$$\therefore\quad \bar{x}=\frac{a^3/3}{\frac{\pi a^2}{4}}=\frac{4a}{3\pi},\; \bar{y}=\frac{b}{2a}\cdot\frac{2}{3}a^3\cdot\frac{4}{\pi a^2}=\frac{4b}{3\pi}.$$

$\therefore \left(\frac{4a}{3\pi}, \frac{4b}{3\pi}\right)$ is the C.G. of the quadrant *OAB*.

REMARK

By putting $a = b$, we find that the C.G. of a quadrant of a uniform circular lamina is $\left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right)$.

Exercise 20.1

1. Particles of masses 1 gm., 2 gms., 3 gms., 4 gms. and 5 gms. are placed at the points

$(4, 0)$, $(3, 4)$, $(-5, 4)$, $(0, -5)$, $(-5, 0)$ respectively. Find the coordinates of their centre of mass.

2. Find the C.G. of a uniform square plate $ABCD$ of weight 10 lbs. together with weights 20, 30, 40, 50 lbs. placed at its four corners A, B, C, D respectively.

3. ABC is a triangle right-angled at A, AB being 12 cms. and AC, 15 cms. Weights proportional to 2, 3, 4 gms. respectively are placed at A, C and B. Find the distances of their centre of gravity from B and C.

4. ABC is an equilateral triangle of side 2 ft. At A, B and C are placed weights proportional to 5, 1 and 3; and at the midpoints of the sides BC, CA and AB weights proportional to 2, 4 and 6. Show that their centre of gravity is at a distance of 16 inches from B.

5. Masses weighing 1, 2, 3, 4 lbs. are placed respectively at the corners A, B, C, D of a rectangle; $AB = 6$ ft. and $BC = 12$ ft. Find the perpendicular distances of the centre of gravity from AB and BC.

6. Find the C.G. of the homogeneous area bounded by the parabola $y^2 = 4ax$ and the double ordinate $x = b$.

7. Find the C.G. of a uniform sector of a circle subtending an angle 2α at the centre.

8. Find the C.G. of a uniform semicircular lamina of radius a.

9. Prove that the C.G. of a unform circular lamina is at its centre.

10. Find the C.G. of the area of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ lying in the first quadrant.

11. Find the C.G. of the area between the sine curve $y = \sin x$ and the x-axis where

$$0 \leqslant x \leqslant \pi.$$

12. Find the C.G. of the area between the cosine curve $y = \cos x$ and the x-axis where

$$-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}.$$

13. Find the C.G. of the area of the parabola $(x/a)^{\frac{1}{2}} + (y/b)^{\frac{1}{2}} = 1$ between the curve and the coordinate axes.

14. Find the C.G. of the area bounded by the parabola $y^2 = 4ax$, the axis of x and the latus rectum.

21 Non-Concurrent Coplanar Forces

In this chapter, we shall investigate the problem of equilibrium of a system of non-concurrent forces acting in a plane.

21.1 Reduction of a system of non-concurrent coplanar forces

A system of coplanar forces acting on a rigid body can be reduced to a single force acting at an arbitrarily chosen point in the plane of the forces together with a couple.

Let a system of non-concurrent coplanar forces P_1, P_2, P_3............act at the points A_1, A_2, A_3................respectively whose Cartesian coordinates are (x_1, y_1), (x_2, y_2), (x_3, y_3)........referred to the rectangular axes $X'OX$, $Y'OY$ in the plane of the forces.

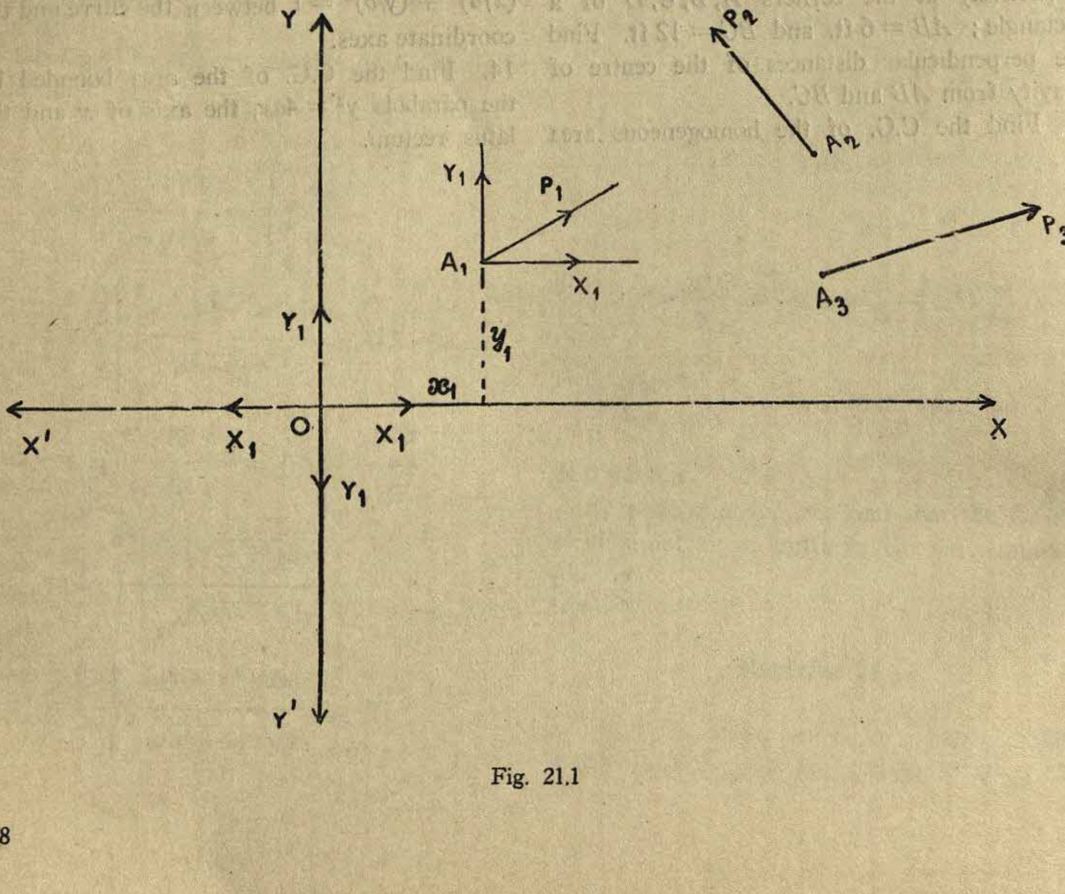


Fig. 21.1

Let us consider the force P_1 whose resolved parts parallel to the axes are X_1, Y_1. At the origin O, let us introduce two pairs of equal and opposite forces X_1, X_1 along $X'OX$ and Y_1, Y_1 along $Y'OY$. These forces form a null system and do not produce any effect on the given system of forces.

Now the forces X_1 at A_1 parallel to OX and X_1 at O along OX' form a couple of moment $-y_1X_1$. Similarly, the forces Y_1 at A_1 parallel to OY and Y_1 at O along OY' form a couple of moment $x_1 Y_1$. Combining these two couples, we have a single equivalent couple $(x_1Y_1 - y_1X_1)$ in the sense of rotation from OX to OY. We are also left with a force X_1 along OX and Y_1 along OY, parallel to their former directions. Proceeding in the same way with each force of the system and combining, we find that the given system of forces reduce to (*i*) a single force R having components

$$X = \Sigma X_1 \text{ along } OX,$$
$$Y = \Sigma Y_1 \text{ along } OY$$

and (*ii*) a single couple of moment

$$G = \Sigma (x_1 Y_1 - y_1 X_1) \tag{1}$$
$$\therefore \quad R = \sqrt{X^2 + Y^2} \tag{2}$$

If the resultant R makes an angle θ with the positive direction of the x-axis, then $X = R\cos\theta$, $Y = R\sin\theta$ so that

$$\theta = \tan^{-1}(Y/X). \tag{3}$$

We thus find that a system of non-concurrent coplanar forces can be reduced to a single resultant R acting at any arbitrary origin O together with a couple of moment G about O.

REMARK

We find that the value of G depends on the origin O, which is called the **base point.** This base point may be taken anywhere on the plane of the forces.

COROLLARY

1. *Conditions of equilibrium*

(*a*) Necessary Conditions : The necessary conditions of equilibrium of the given system of forces are

$$R = 0,\ G = 0;$$
$$\text{i.e. } X = 0,\ Y = 0,\ G = 0. \tag{4}$$

(*b*) Sufficient Conditions : The conditions (4) are also sufficient because $X = 0$, $Y = 0$ mean $R = 0$ so that there cannot be any translatory motion ; also $G = 0$ means there cannot be any rotation.

Hence when the conditions (4) are satisfied, the given system of forces must be in equilibrium.

2. When the system of forces is not in equilibrium, it reduces to a single couple G if $R = 0$ and to a single force R if $G = 0$.

If neither R nor G is zero, the system reduces to a single force because a couple and a force in the same plane are together equivalent to a single force.

3. *Moment of the resultant couple about any point* : Let G' be the moment of the resultant couple about O' whose coordinates are (ξ, η) referred to O as origin. We now shift the origin from O to O', keeping the directions of the axes unaltered. Let (x_1', y_1') be the coordinates of A_1 relative to O' as origin.

Then $x_1 = x_1' + \xi,\ y_1 = y_1' + \eta$.

$\therefore$ Moment of the resultant couple about the new origin O' is

$$\begin{aligned} G' &= \Sigma\,[x_1' Y_1 - y_1' X_1] \\ &= \Sigma\,[\,(x_1 - \xi)Y_1 - (y_1 - \eta)X_1] \\ &= \Sigma\,[(x_1Y_1 - y_1X_1) - \xi Y_1 + \eta X_1] \\ &= \Sigma\,[x_1Y_1 - y_1X_1] - \xi\,\Sigma Y_1 + \eta\,\Sigma X_1 \\ &= G - \xi Y + \eta X. \end{aligned}$$

4. *Equation of the line of action of the resultant* : If O' (ξ, η) be a point on the line of action of the resultant R to which the system reduces, then the algebraic sum of moments of the forces about O'

$$= \text{moment of } R \text{ about } O'$$
$$= O\,;\ \text{i.e. } G' = 0.$$
$$\therefore \quad G - \xi Y + \eta X = 0. \tag{5}$$

If we regard (ξ, η) as the current coordinates, then equation (5) represents the equation of the line of action of the resultant R.

21.2 Illustrative examples

EXAMPLE 1

Forces P, Q, R act respectively along the lines $x = o$, $y = o$, $x \cos\theta + y \sin\theta = p$.

Find the magnitude of the resultant force and the equation to the line of action of the resultant.

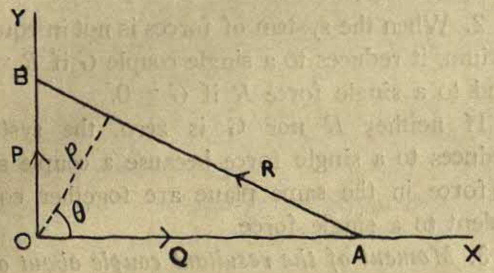


Fig. 21.2

Let X, Y be the total resolved parts of the forces parallel to the rectangular axes OX and OY respectively.

∴ By resolving the forces along the axes, we have

$$X = Q - R\cos(90^\circ - \theta) = Q - R\sin\theta,$$

$$Y = P + R\sin(90^\circ - \theta) = P + R\cos\theta.$$

∴ Magnitude of the resultant

$$= \sqrt{X^2 + Y^2} = \sqrt{P^2 + Q^2 + R^2 - 2QR\sin\theta + 2PR\cos\theta}.$$

Moment of the forces about O is

$$G = Q \cdot 0 + P \cdot 0 + R \cdot p = Rp.$$

∴ Equation to the line of action of the resultant is

$$G - xY + yX = 0$$

or, $pR - x(P + R\cos\theta) + y(Q - R\sin\theta) = 0$

or, $Px - Qy + R(x\cos\theta + y\sin\theta - p) = 0.$

EXAMPLE 2

The moments of a system of coplanar forces acting in the (x, y)-plane about (o, o), (a, o), (o, a) are aw, $2aw$, $3aw$ respectively. Find the components parallel to the coordinate axes and the line of action of the single force to which the system reduces.

Let X, Y be the total resolved parts of the forces parallel to the x and y-axes respectively.

Taking moments about (o, o), we have $aw = G - o \cdot Y + o \cdot X = G$;

$$\therefore \quad G = aw \qquad (1)$$

Taking moments about (a, o),

$$2aw = aw - aY; \qquad \therefore \quad Y = -w \qquad (2)$$

Taking moments about (o, a),

$$3aw = aw + aX; \qquad \therefore \quad X = 2w \qquad (3)$$

∴ Equation to the line of action of the resultant is

$$G - xY + yX = 0,$$

or, $aw + xw + 2wy = 0,$

or, $x + 2y + a = 0. \qquad (4)$

Component of the resultant parallel to the x-axis is $X = 2w$ and component of the resultant parallel to the y-axis is $Y = -w$.

EXAMPLE 3

A system of coplanar forces is such that the sum of their moments about any point in a given straight line is always the same. State what inferences may be drawn about the resultant of the system.

Let us take the given straight line as the axis of x, any point O on it being taken as the

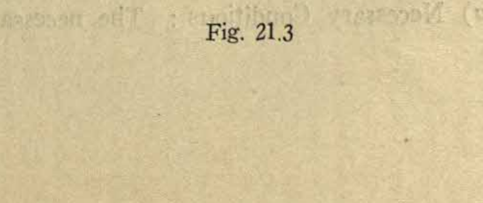


Fig. 21.3

origin. We consider any two points (x_1, o), (x_2, o) on it.

Taking moments about (x_1, o), we have

$$G' = G - x_1 Y.$$

Taking moments about (x_2, o), we have

$$G'' = G - x_2 Y.$$

By the given condition $G' = G''$

or, $G - x_1 Y = G - x_2 Y$

or, $(x_1 - x_2) Y = o$;

$$\therefore \quad Y = o \, (\because \; x_1 \neq x_2).$$

If R be the resultant of the system,

$$R^2 = X^2 + Y^2 = X^2 \; (\because \; Y = o).$$

$\therefore$ R is either zero (when $X = o$) or is parallel to OX.

Hence the resultant is either zero or parallel to the given line.

Exercise 21.1

1. Three forces P, Q, R act along the sides of the triangle formed by the lines $x + y = 1$, $y - x = 1$, $y = 2$ respectively.

Find the equation of the line of action of the resultant.

2. The algebraic sums of moments of a system of coplanar forces about points whose coordinates are $(1, 0)$, $(0, 2)$ and $(2, 3)$ referred to rectangular axes are H, $2H$ and $3H$ respectively. Find the tangent of the angle which the direction of the resultant force makes with the x-axis.

3. The moments of a system of coplanar forces about the points $(0, 0)$, $(10, 0)$, $(0, 7)$ are 184, -46, 65 ft. lbs. respectively. Find where the system meets the axis of x and find its components parallel to the axes.

4. $ABCD$ is a square whose side is 2 units in length. Forces a, b, c, d act along the sides AB, BC, CD, DA, taken in order, and $p\sqrt{2}$, $q\sqrt{2}$ act along AC and DB respectively. Show that if $p + q = c - a$ and $p - q = d - b$, the forces are equivalent to a couple of moment $a + b + c + d$.

5. A system of coplanar forces has the total moments H, $2H$ respectively about points whose coordinates are $(2a, o)$, (o, a) referred to fixed rectangular axes. The total resolved parts of the forces along the line $y = x$ vanishes. Find where the line of action of the resultant meets the axes of coordinates.

6. The straight line $4x + 3y = 5$ meets the coordinate axes ox and oy at the points A and B respectively. If forces P, Q, R act along the lines OB, OA and AB, find the magnitude of the resultant and the equation to its line of action.

7. Find the locus of the points about which the algebraic sum of the moments of a system of coplanar forces is constant.

8. The algebraic sum of the moments of a system of coplanar forces about each of three non-collinear points A, B, C is zero. Show that the system is in equilibrium.

9. Moments of the resultant R of a system of coplanar forces are G, $G + J_1$, $G + J_2$ respectively about O, A, B whose Cartesian coordinates are $(0, 0)$, $(r_1 \cos\theta_1, r_1 \sin\theta_1)$ and $(r_2 \cos\theta_2, r_2 \sin\theta_2)$ referred to 0 as origin. Show that

$$R^2 \sin^2(\theta_1 - \theta_2) = \frac{J_1^2}{r_1^2} + \frac{J_2^2}{r_2^2} - 2\frac{J_1 J_2 \cos(\theta_1 - \theta_2)}{r_1 r_2}.$$

10. An equilateral triangular lamina ABC resting on a smooth horizontal plane is acted upon by a force of 5 lbs. wt. along BC, 3 lbs. wt. along AC and 2 lbs. wt. along AD, where AD is perpendicular to BC. Find the force at B and the couple which will keep the lamina at rest.

11. Prove that in all cases a system of coplanar forces can be reduced to two forces, one of which acts through a given point and the other along a given straight line.

22 Graphical Statics

In Chapter 16, we have obtained the resultant of a system of concurrent coplanar forces by analytical methods. In Chapter 21, we have made use of analytical methods to determine the resultant of a system of non-concurrent coplanar forces. We shall now show how the resultant of any number of coplanar forces can be determined **graphically.**

22.1 Resultant of a system of coplanar forces acting at a point

The resultant of a system of coplanar forces acting at a point may be determined graphically with the help of the Polygon of Forces.

The underlying principle of the Polygon of Forces is that 'forces acting at a point *O* and represented in magnitude and direction by the sides (taken in order) of the polygon *ABCDEF* are in equilibrium'. This implies that the resultant of forces represented by *AB*, *BC*, *CD*, *DE*, *EF* must be equal and opposite to the force represented by *FA*, i.e. the resultant of forces represented by *AB*, *BC*, *CD*, *DE*, *EF* is represented by *AF* in magnitude and direction.

By employing this method, we can find the resultant of any number of coplanar forces acting at a point. We call this method to be a **graphical method.** Here the forces are represented in magnitude and direction by line-segments which are the sides of the polygon. The polygon so obtained is called a **force diagram.** Hence the method of solving problems in statics by representing the forces in magnitude and direction by line-segments is known as the **graphical method.**

22.2 To find graphically the resultant of any number of non-concurrent coplanar forces

Let the given forces be P, Q, R, S, T whose lines of action are as shown in Fig. 22.1. We have to find out the resultant of these forces by graphical methods.

Starting from any point A_1 in Fig. 22.2 we draw line-segments A_1B_1, B_1C_1, C_1D_1, D_1E_1, E_1F_1 to represent P, Q, R, S, T respectively in magnitude and direction. F_1A_1 is joined to complete the polygon. We shall now show that the resultant force is represented by A_1F_1 in magnitude and direction.

Let us take any point O in the plane of the figure $A_1B_1C_1D_1E_1F_1$ and join it to the cor-

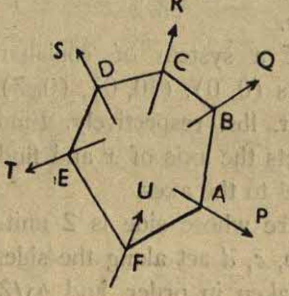


Fig. 22.1

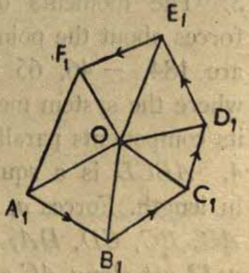


Fig. 22.2

ners A_1, B_1, etc. The polygon in Fig. 22.2 is called the **force diagram** and O is called its **pole.**

Let us take any point A on the line of action of P. A polygon is then constructed by following the procedure given below :

AB is drawn parallel to B_1O
to meet the line of action of Q at B,
BC is drawn parallel to C_1O
to meet the line of action of R at C,
CD is drawn parallel to D_1O
to meet the line of action of S at D,
DE is drawn parallel to E_1O
to meet the line of action of T at E ;

then a line parallel to F_1O is drawn through E and another line parallel to OA_1 is drawn through A ; these two lines meet at (say) F.

We shall prove that the resultant force is represented in magnitude and direction by A_1F_1 and it actually acts through F.

The proof is as follows :

The force P acting at $A = \overline{A_1B_1} = \overline{A_1O}$ along $FA + \overline{OB_1}$ along BA.
The force Q acting at $B = \overline{B_1C_1} = \overline{B_1O}$ along $AB + \overline{OC_1}$ along CB.
The force R acting at $C = \overline{C_1D_1} = \overline{C_1O}$ along $BC + \overline{OD_1}$ along DC.
The force S acting at $D = \overline{D_1E_1} = \overline{D_1O}$ along $CD + \overline{OE_1}$ along ED.
The force T acting at $E = \overline{E_1F_1} = \overline{E_1O}$ along $DE + \overline{OF_1}$ along FE.

By combining these results, we find that the resultant of the forces P, Q, R, S, T

$= \overline{A_1O}$ along $FA + \overline{OF_1}$ along FE
$=$ a force represented by A_1F_1 acting through F, the point of intersection of FA and FE.

The figure $A_1B_1C_1D_1E_1F_1$ is called the **Force Polygon** for the given system, while the figure $ABCDEF$ is called a **Link** or **Funicular Polygon** (or Rope Polygon).

REMARKS

1. By varying the positions of A and O, the shape of the funicular polygon can be varied in a number of ways, but the point of intersection of FA and FE will always lie on the same straight line, the line of action of the resultant.

2. If the point F_1 of the force polygon coincides with A_1 of the polygon, the polygon is said to be closed. This means that the resultant force vanishes. We cannot, however, say that the given forces are in equilibrium. Since A_1 and F_1 are coincident, the line through E parallel to F_1O and the line through A parallel to A_1O are either **coincident** or **parallel.** In the former case the force system reduces to two equal and opposite forces along the same straight line and the system is in equilibrium. In the latter case the system reduces to a couple of equal and opposite forces A_1O and OF_1 in parallel lines and the moment of the couple is A_1OX (the distance between these parallel lines).

From the above discussions, we see that if EF and FA are coincident, i.e. EFA is a straight line, the equal and opposite forces along them balance each other and the system is in equilibrium. In this case the funicular polygon is a closed one. The arguments here are equally applicable to every funicular polygon that can be drawn.

We may, therefore, conclude that the conditions of equilibrium are

(i) The Force Polygon of the system must be closed.
(ii) Any Funicular Polygon of the system must be closed.
(iii) If any one funicular of the system is closed, every funicular of the system is closed.

(iv) If the system is equivalent to a couple, the force polygon is closed, and the first and last sides of all funiculars are parallel.

3. It is evident that, the pole O being arbitrary, there can be an infinite number of such funicular polygons.

22.3 To find graphically the resultant of any number of parallel forces

The resultant of a system of parallel forces can be found graphically by following the same procedure as described in Section 22.2 above; the only difference in this case is that the sides of the force diagram all lie in the same straight line.

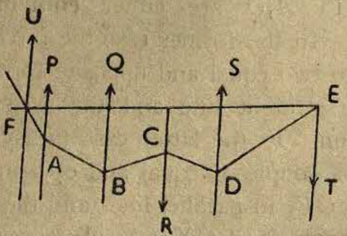


Fig 22.3

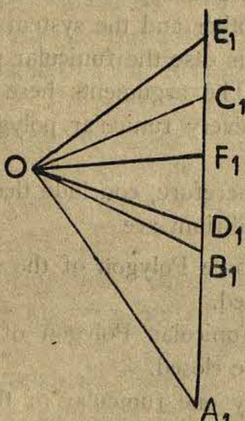


Fig. 22.4

Let P, Q, R, S, T be five parallel forces acting along the given lines as shown in Fig. 22.3. Let us now mark off A_1B_1, B_1C_1, C_1D_1, D_1E_1, E_1F_1 on a suitable scale, to represent P, Q, R, S, T taken in order. We take any point O and join it to A_1, B_1, C_1, D_1, E_1, F_1. Next we take any point A on the line of action of P and proceed as follows :

AB is drawn parallel to OB_1
to meet the line of action of Q at B,

BC is drawn parallel to OC_1
to meet the line of action of R at C,

CD is drawn parallel to OD_1
to meet the line of action of S at D,

DE is drawn parallel to OE_1
to meet the line of action of T at E ;

then a line parallel to F_1O is drawn through E and another line parallel to A_1O is drawn through A ; these two lines meet at F.

We shall prove that the resultant force U is represented in magnitude and direction by A_1F_1 and it actually acts through F.

The proof is as follows :

The force P acting at $A = \overline{A_1B_1} = \overline{A_1O}$ along $AF + \overline{OB_1}$ along AB.

The force Q acting at $B = \overline{B_1C_1} = \overline{B_1O}$ along $BA + \overline{OC_1}$ along BC.

The force R acting at $C = \overline{C_1D_1} = \overline{C_1O}$ along $CB + \overline{OD_1}$ along CD.

The force S acting at $D = \overline{D_1E_1} = \overline{D_1O}$ along $DC + \overline{OE_1}$ along DE.

The force T acting at $E = \overline{E_1F_1} = \overline{E_1O}$ along $ED + \overline{OF_1}$ along FE.

By combining these results, we find that the resultant of the forces P, Q, R, S, T

$= \overline{A_1O}$ along $AF + \overline{OF_1}$ along FE

$= \overline{A_1F_1}$ along a line through F parallel to A_1F_1

= a force represented in magnitude and direction by $\overline{A_1F_1}$ and acting through F, the point of intersection of AF and EF.

The figure $A_1B_1C_1D_1E_1F_1$ is called the Force Polygon for the given system of parallel forces, while the figure $ABCDEF$ is called the Funicular Polygon.

22.4 To find the magnitudes of two parallel forces which act along given straight lines and are together equivalent to a given system of parallel forces.

Let P, Q, R be the given parallel forces, and AB, CD the given straight lines. Although we consider only three forces, the method is a general one and can equally be applied to any number of parallel forces.

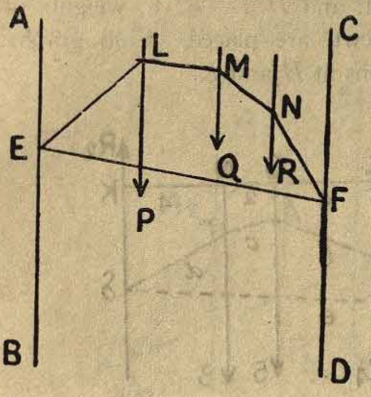


Fig. 22.5

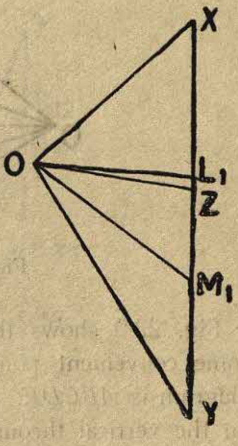


Fig. 22.6

We draw XL_1, L_1M_1, M_1Y to represent the forces P, Q, R taken in order. Next we take any point O and join it to X, L_1, M_1, Y. Starting with a point E on AB, we proceed as follows :

EL is drawn parallel to OX to meet P at L ;
LM „ „ „ „ OL_1 „ „ Q at M ;
MN „ „ „ „ OM_1 „ „ R at N ;
NF „ „ „ „ OY „ „ CD at F.

Now EF is joined and OZ is drawn parallel to EF to meet XY at Z. Then we have

force P at L = $\overline{XL_1}$ = $\overline{XO}$ along LE + $\overline{OL_1}$ along LM ;
force Q at M = $\overline{L_1M_1}$ = $\overline{L_1O}$ along ML + $\overline{OM_1}$ along MN ;
force R at N = $\overline{M_1Y}$ = $\overline{M_1O}$ along NM + $\overline{OY}$ along NF.

The forces along each of the lines LM and MN being equal and opposite balance themselves. Thus we are left with forces :

$\overline{XO}$ along LE and $\overline{OY}$ along NF.

We may suppose that $\overline{XO}$ acts at E and $\overline{OY}$ acts at F.

Now $\overline{XO}$ acting at E = $\overline{XZ}$ along AB + $\overline{ZO}$ along FE
and $\overline{OY}$ acting at F = $\overline{OZ}$ along EF + $\overline{ZY}$ along CD.

We find that the forces along EF are equal and opposite. The system is, therefore, equivalent to (1) a force $\overline{XZ}$ along the given line AB and (2) a force $\overline{ZY}$ along the given line CD.

REMARK

When a beam is supported at two points and carries loads at other points, the supporting forces on the beam can be graphically determined with the help of the method described above.

22.5 **Illustrative examples**

EXAMPLE 1

ABC is a triangle whose sides *AB, BC, CA* are respectively 12, 10 and 15 cms. long and *BD* is the perpendicular from *B* on *CA*. Find graphically the magnitude and the line of action of the resultant of forces 8 from *A* to *C*, 4 from *C* to *B*, 3 from *B* to *A* and 2 from *B* to *D*.

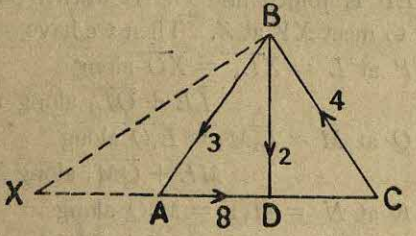


Fig. 22.7

The forces act in given lines as shown in Fig. 22.7. We now draw the force-polygon as shown in Fig. 22.8 so that *OE, EF, FG, GH* represent the forces 4, 2, 3, 8 respectively. The force-polygon is then closed by joining *OH*; hence *OH* represents the resultant of the given system of forces. By actual measurement, this is found to be equal to 3.4.

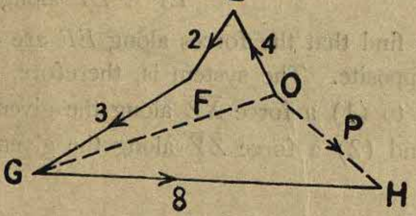


Fig. 22.8

Let θ be the angle which *HO* makes with *GH*, i.e. with *AC*. By actual measurement, $\theta = 13°$ approximately.

The three forces 4, 2, 3 meet at *B* and hence their resultant passes through *B*. Therefore, the resultant of these three forces is represented by *OG* in magnitude and direction. We now draw a line through *B* parallel to *OG* to meet *CA* (produced) at *X*. The resultant of the four forces must, therefore, pass through *X*.

EXAMPLE 2

A uniform beam *HK*, of length 12 ft. and weight 5 cwt., is supported at *H* and *K* so as to be horizontal; at points *L* and *M* on it such that *HL* = 2 ft. and *HM* = 8 ft., weights of 4 cwt. and 3 cwt. are placed. Find graphically the reactions at *H* and *K*.

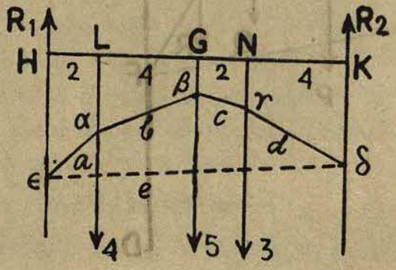


Fig. 22.9

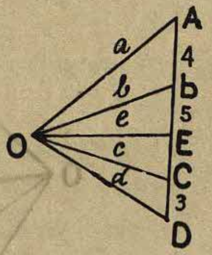


Fig. 22.10

Fig. 22.9 shows the loaded beam. Taking some convenient point *O* as pole, the force-diagram is *ABCDE*. Starting with any point α on the vertical through *L*, we proceed as follows : $\alpha\varepsilon$ is drawn parallel to *OA* to meet the vertical through H at ε ; $\alpha\beta$ is drawn parallel to OB to meet the vertical through G at β ; $\beta\gamma$ is drawn parallel to OC to meet the vertical through M at γ ; $\gamma\delta$ is drawn parallel to OD to meet the vertical through K at δ. $\delta\varepsilon$ is joined and OE is drawn through O parallel to $\delta\varepsilon$ to meet AD at E. Then DE represents R_1 and EA represents R_2 on the same scale according to which $AB = 4$ cwt. By actual measurement, we then have

$$R_1 = 6.83 \text{ cwt. and } R_2 = 5.17 \text{ cwt.}$$

EXAMPLE 3

Given a system of parallel forces P_1, P_2, P_3, P_4, P_5, construct a funicular polygon which passes through two given points X on the line of action of P_1 and Y on the line of action of P_4.

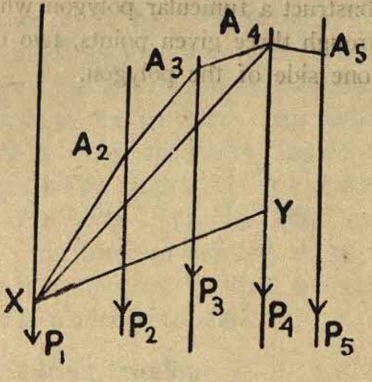


Fig. 22.11

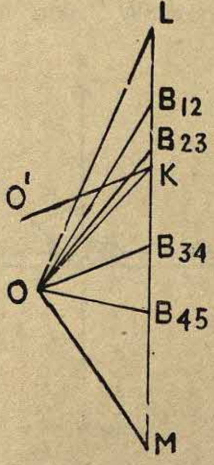


Fig. 22.12

Taking any point O as pole, a force diagram LM (as shown in Fig. 22.12) is drawn for the given forces. Starting from the point X on the line of action of P_1, we draw consecutive sides of a funicular polygon $XA_2A_3A_4A_5$. Through O, a line OK parallel to XA_4 is drawn to meet LM in K.

The forces P_1, P_2, P_3, P_4 are then equivalent to (1) a force LK at X and (2) a force KB_{45} at A_4.

Now KO' is drawn parallel to YX.

If we now construct a funicular polygon using any point O' on KO' as pole starting from X, it will pass through Y, because $O'K$ being parallel to XY, the four forces P_1, P_2, P_3, P_4 are equivalent to (1) a force LK at X and (2) a force KB_{45} at Y.

Exercise 22.1

1. Forces 1, 3, −4 lbs. wt. act in order round the sides of an equilateral triangle ABC of side 2 inches drawn on a rigid lamina. Give a graphical construction for the magnitude, direction and position of their resultant.
2. Four forces acting in one plane are in equilibrium ; their lines of action are all given and also the magnitude of one of them. Give a geometrical construction to find the magnitudes of the others.
3. $ABCDEF$ is a regular hexagon. Show that the forces which must act along AC, AF and DE to produce equilibrium with a force of 40 lbs.-wt. acting along EC are respectively 10, 17.32 and 34.64 lbs.-wt.
4. A light rod AB, 10 ft. long, is supported at A and at another point. A load of 1 lb. is suspended from B, and loads of 5 lbs. and 2 lbs. at points 2 ft. and 6 ft. from A. If the second

support is at the middle point of AB, find graphically the pressures on the supports.

If the pressure on the support at A is required to be 4 lbs. wt., find graphically where the second point of support must be.

5. A horizontal beam 20 ft. long is supported at its ends and carries loads of 3, 2, 5 and 4 cwt. at distances of 3, 7, 12 and 15 ft. respectively from one end. Find by means of a funicular polygon the thrusts on the two ends.

6. A uniform plank of weight 40 lbs. and length 8 ft. carries a weight of 12 lbs. at a point 3 ft. from one end. If the plank is supported at its ends in a horizontal position, determine graphically the pressures on the supports.

7. Weights of 5, 10, 12, 8 and 6 lbs. rest upon a beam at distances 1, 4, 7, 9 and 12 ft. from one end. The beam is supported at distances 5 and 15 ft. from the same end. Find graphically the supporting forces.

8. $ABCDE$ is a light rod supported horizontally at B and D, and carrying loads of 10, 15 and 20 lbs. at A, C and E where $AB = DE = 2$ ft., $BC = 3$ ft. and $CD = 4$ ft. Determine graphically the pressures on the supports.

9. Of four coplanar forces in equilibrium, given the magnitude of one and the lines of action of all. Show how to find graphically the magnitudes of all the forces.

10. For any system of coplanar forces, construct a funicular polygon three of whose sides shall pass each through a given point.

11. Construct a funicular polygon which shall pass through three given points, two of which lie on one side of the polygon.

23 Friction

When two **perfectly smooth** bodies are in contact with one another, the mutual action between them acts at the point of contact along the common normal to the surfaces in contact. In reality, it is actually hypothetical to say that the surface of a body is 'perfectly smooth', however much its surface may be polished. All bodies are capable of exerting some resistance which tends to prevent them from sliding over one another. This resistance created between two rough bodies which tries to prevent sliding of one body over the other is called **friction**; the force of friction always acts in the direction opposite to that in which the body has a tendency to slide.

23.1 Limiting friction

Let us consider a body of weight W placed on a horizontal table. When the body is at rest, its weight W and the normal reaction R of the table on it maintain equilibrium.

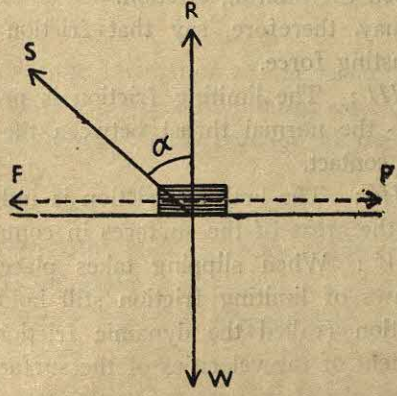


Fig. 23.1

If the body is now pulled, with the help of a string tied to it, by a horizontal force P, the surface of the table develops a frictional force F in a direction opposite to that of P.

Let S be the resultant of R and F acting at an angle α to R.

Then $R = S \cos \alpha, F = S \sin \alpha, \tan \alpha = \frac{F}{R}$.

It will be found that, unless P exceeds a certain amount, the body does not move on the table. Since the body remains in equilibrium, we must have $P = F$ and $R = W$.

If we go on increasing the pull P, the force of friction F also goes on increasing and we shall at last reach a limit beyond which it is impossible to increase P further without making the body move.

At this limiting stage the body is just on the point of moving and is said to be in **limiting equilibrium.** The amount of friction that is called into play at the stage of limiting equilibrium is known as **limiting friction.** If the pull P be increased beyond this limit, the friction is no longer able to counteract the pull of the string and the body begins to move. Hence we conclude :

The greatest amount of friction that can be called into play between two surfaces in contact is called the limiting friction.

23.2 Some definitions

23.2 (*a*) *Coefficient of friction*

The ratio of the limiting friction to the normal reaction is known as the coefficient of friction and it is denoted by μ.

Thus $$\mu = \frac{F_{\max}}{R} \qquad (1)$$

where $F_{\max}$ stands for the maximum force of friction.

23.2(*b*) *Angle of friction*

The angle whose tangent is μ, the coefficient of friction is called the angle of friction. Symbolically,

$$\textbf{angle of friction } \lambda = \tan^{-1}\mu \qquad (2)$$

The angle of friction λ may also be defined as the angle made by the resultant of the limiting friction and the normal reaction with the latter.

Thus $\alpha \to \lambda$ as $F \to F_{\max}$ in Fig. 23.1.

From (1) and (2), $\tan \lambda = \mu = \dfrac{F_{\max}}{R}$

$$\text{so that } F_{\max} = \mu R. \qquad (3)$$

23.2(*c*) *Total reaction*

The total reaction is the resultant of the normal reaction R and the frictional force F whether the friction is limiting or not. Thus S is the total reaction in Fig. 23.1.

23.2(*d*) *Cone of friction*

When two bodies are in contact, the cone described with the point of contact as vertex, the common normal as axis and λ (the angle of friction) as the semi-vertical angle is called the cone of friction.

The total reaction S may have any direction lying within or upon the cone but it cannot lie without the cone. In case of limiting equilibrium, S lies on the surface of the cone and in all other situations S lies within the cone.

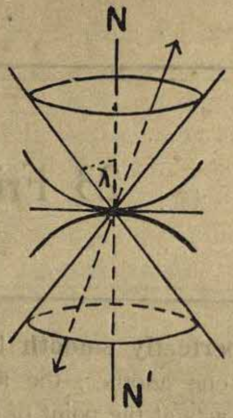


Fig. 23.2

23.3 Laws of friction

The Laws of Friction do not admit of any theoretical proof but have to be found by experiment. The laws are enumerated below :

Law I : When two bodies in contact slide or tend to slide over one another, a force of friction develops at the point of contact in a direction opposite to that in which the point tends to start moving.

Law II : The amount of friction called into play between two surfaces in contact will just suffice to prevent them from sliding on one another, provided that this amount does not exceed the limiting friction.

We may, therefore, say that friction is a self-adjusting force.

Law III : The limiting friction is proportional to the normal thrust between the surfaces in contact.

Law IV : The limiting friction is independent of the areas of the surfaces in contact.

Law V : When slipping takes place, the above laws of limiting friction still hold and the friction (called the dynamic friction) is independent of the velocities of the surfaces in contact.

Law VI : The friction between two bodies varies according to the nature of the material forming their surfaces.

23.4 Friction on inclined planes

The greatest inclination at which a body will remain at rest on an inclined plane is equal to the angle of friction (general theorem).

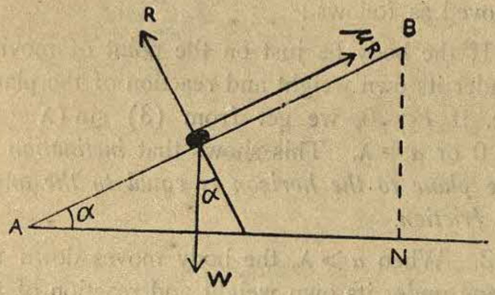


Fig. 23.3

Let a body of weight W be placed on a flat slab and let the slab be gradually raised on one side so as to form an inclined plane. As the slab is raised, the body at first continues to be at rest on the plane as it is prevented from sliding down by the friction of the plane. A stage is then reached when the body is just on the point of sliding down the plane, and any further increase in the inclination will cause the body to slip down. At this particular inclination the body is in limiting equilibrium [see Fig. 23.3].

Let α be the inclination of the plane to the horizon at this position of limiting equilibrium and let R be the normal reaction of the plane on the body.

The frictional force is μR and it acts up the plane because the body is on the point of sliding down the plane.

Resolving along and perpendicular to the plane, we have

$$W\cos(90^\circ-\alpha)=\mu R\text{ ; or, } W\sin\alpha=\mu R$$
$$\text{and}\quad W\cos\alpha=R.$$

$$\therefore\ \tan\alpha=\frac{W\sin\alpha}{W\cos\alpha}=\frac{\mu R}{R}=\mu=\tan\lambda$$

whence λ is the angle of friction.

$$\therefore\quad \alpha=\lambda.$$

whence λ is the angle of friction.

23.4(a) *To find the least force required to pull a body up or down a rough inclined plane*

Let W be the weight of the body, R the normal reaction of the plane on the body and α the inclination of the plane to the horizon.

We shall consider the following two cases.

CASE I : *Let the body be just on the point of sliding down the plane.*

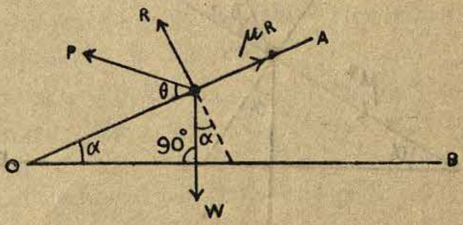


Fig. 23.4

Since the body is on the point of motion down the plane, the frictional force acts up the plane and is equal to μR, μ being the coefficient of friction.

Let a force P be applied to the body at an angle θ with the inclined plane in order to make it move down the plane.

Resolving along and perpendicular to the plane, we get

$$P\cos\theta+W\sin\alpha=\mu R \qquad (1)$$
$$\text{and } P\sin\theta+R=W\cos\alpha \qquad (2)$$

Eliminating R from (1) and (2), we get

$$P\cos\theta+W\sin\alpha=\mu(W\cos\alpha-P\sin\theta)$$
$$\text{or, }\ P(\cos\theta+\mu\sin\theta)=W(\mu\cos\alpha-\sin\alpha)$$
$$\text{or, }\ P(\cos\theta+\tan\lambda\sin\theta)$$
$$=W(\tan\lambda\cos\alpha-\sin\alpha)$$

[$\because$ $\mu=\tan\lambda$, λ being the angle of friction]

$$\text{or,}\quad \frac{P(\cos\theta\cos\lambda + \sin\theta\sin\lambda)}{\cos\lambda} = \frac{W(\sin\lambda\cos\alpha - \sin\alpha\cos\lambda)}{\cos\lambda}$$

$$\text{or,}\quad P\cos(\theta - \lambda) = W\sin(\lambda - \alpha)$$

$$\therefore\quad P = W\frac{\sin(\lambda - \alpha)}{\cos(\theta - \lambda)} \qquad (3)$$

The force P is least when $\cos(\theta - \lambda) = 1$ or $\theta = \lambda$, and then its magnitude is $W\sin(\lambda - \alpha)$ where $\alpha < \lambda$.

CASE II : *Let the body be just on the point of sliding up the plane.*

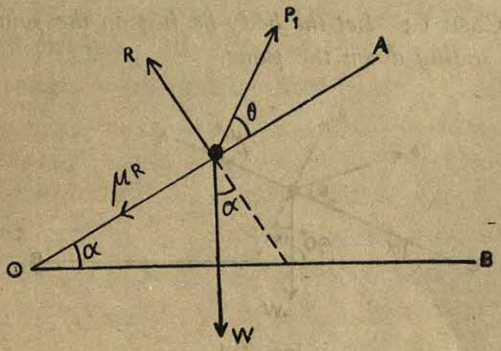


Fig. 23.5

Since the body is on the point of motion up the plane, the friction acts down the plane and is equal to μR.

Let a force P_1 be applied to the body at an angle θ with the plane in order to make it move up the plane.

Resolving along and perpendicular to the plane, we get

$$\mu R + W\sin\alpha = P_1\cos\theta \qquad (4)$$

$$\text{and } W\cos\alpha = R + P_1\sin\theta \qquad (5)$$

Eliminating R from (4) and (5), we have

$$P_1\cos\theta - W\sin\alpha = \mu(W\cos\alpha - P_1\sin\theta)$$

$$\text{or,}\quad P_1(\cos\theta + \mu\sin\theta) = W(\mu\cos\alpha + \sin\alpha)$$

$$\text{or,}\quad P_1(\cos\theta + \tan\lambda\sin\theta) = W(\tan\lambda\cos\alpha + \sin\alpha)$$

$$\text{or,}\quad P_1\frac{\cos(\theta - \lambda)}{\cos\lambda} = W\frac{\sin(\lambda + \alpha)}{\cos\lambda}$$

$$\therefore\quad P_1 = W\frac{\sin(\lambda + \alpha)}{\cos(\theta - \lambda)} \qquad (6)$$

We find from (6) that the force P_1 is least when $\theta = \lambda$, and then its magnitude is $W\sin(\lambda + \alpha)$.

REMARKS

1. The theorem of Section 23.4 may also be proved as follows :

If the body be just on the point of moving under its own weight and reaction of the plane, i.e. if $P = 0$, we get from (3) $\sin(\lambda - \alpha) = 0$ or $\alpha = \lambda$. This shows that *inclination of* the *plane to the horizon is equal to the angle of friction.*

2. When $\alpha > \lambda$, the body moves down the plane under its own weight and reaction of the plane. The question of finding the least force in this case becomes, therefore, redundant.

3. If $\theta = 0$, the limiting values of P and P_1 for which the body is to remain in equilibrium, are respectively $W\sin(\lambda - \alpha)\sec\lambda$ and $W\sin(\lambda + \alpha)\sec\lambda$. If the value of the impressed force lies between these two limiting values of P and P_1, the body remains in equilibrium but is not on the point of moving up or down the plane, i.e. the **equilibrium is non-limiting.**

23.5 Illustrative examples

EXAMPLE 1

A uniform rod rests on two rough pegs A and B. AB is inclined to the horizon at an angle α, and the coefficients of friction at A and B are μ_1, μ_2 respectively. If the rod is in limiting equilibrium and the centre of the rod divides AB into two portions of lengths a and b, prove that

$$\tan\alpha = \frac{b\mu_1 + a\mu_2}{a + b}$$

$$\text{and}\quad a : b = (\mu_1 - \tan\alpha) : (\tan\alpha - \mu_2)$$

Let G be the centre of the rod CD which rests on the pegs A and B.

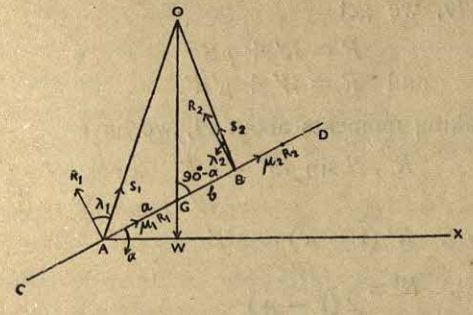


Fig. 23.6

It is given that $AG = a$, $GB = b$.

Let S_1, S_2 be the resultant of the normal reaction and the force of friction at A and B respectively. The lines of action of S_1 and S_2 meet at O. For limiting equilibrium, the vertical line through G must pass through O.

We know from Trigonometry that

$$AB \cot OGB$$
$$= GB \cot OAG - AG \cot OBG$$

or, $$(a + b) \cot (90° - \alpha)$$
$$= b \cot(90° - \lambda_1) - a \cot(90° + \lambda_2)$$

or, $$(a + b) \tan \alpha = b \tan \lambda_1 + a \tan \lambda_2$$
$$= b\,\mu_1 + a\,\mu_2 \qquad (1)$$

$$\therefore \quad \tan \alpha = \frac{b\mu_1 + a\mu_2}{a + b}.$$

Again, from (1)

$$a(\tan \alpha - \mu_2) = b(\mu_1 - \tan \alpha)$$

$$\therefore \quad a : b = (\mu_1 - \tan \alpha) : (\tan \alpha - \mu_2).$$

Hence the result.

EXAMPLE 2

Two equal uniform rods AO and BO are freely jointed at O and rest in a vertical plane with the ends A and B in contact with a rough horizontal plane. If the equilibrium is limiting and the coefficient of friction be μ, show that

$$\sin AOB = \frac{4\mu}{1 + 4\mu^2}.$$

Let W be the weight of each rod and G, G' their centres of gravity. OC is drawn perpendicular to the horizontal plane to meet AB at C.

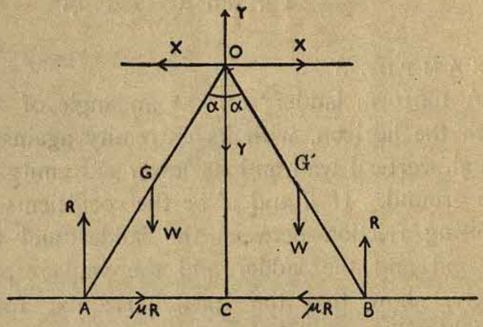


Fig. 23.7

The reactions at the joint O on either rod are as shown in Fig. 23.7.

At A and B, the rods have a tendency to slip outwards ; the friction, therefore, acts inwards.

Let $\angle AOC = \angle BOC = \alpha$.

Let us first consider the rod AO.

Resolving the forces on the rod AO along horizontal and vertical directions, we get

$$X = \mu R \text{ and } Y + R = W. \qquad (1)$$

Considering the rod BO, and resolving the forces on the rod BO horizontally and vertically, we get

$$X = \mu R \text{ and } R - Y = W. \qquad (2)$$

From (1) and (2), we get

$$Y = O \text{ and } R = W. \qquad (3)$$

Taking moments about O, of the forces acting on the rod AO, we have

$$W.OG \sin \alpha + \mu R.AO \cos \alpha$$
$$- R.AO \sin \alpha = 0$$

or, $$Wa \sin \alpha + 2\mu Ra \cos \alpha - 2Ra \sin \alpha = 0$$

(if $2a$ be the length of each rod)

or, $$W \sin \alpha + 2\mu W \cos \alpha$$
$$- 2W \sin \alpha = 0 \quad \text{by (3)}$$

or, $$2\mu \cos \alpha = \sin \alpha$$

or, $$\tan \alpha = 2\mu.$$

$\therefore \quad \sin AOB = \sin 2\alpha$

$$= \frac{2 \tan \alpha}{1 + \tan^2\alpha} = \frac{4\mu}{1 + 4\mu^2}.$$

EXAMPLE 3

A uniform ladder rests at an angle of 45° with the horizon, with its extremity against a rough vertical wall and its lower extremity on the ground. If μ and μ' be the coefficients of limiting friction between the ladder and the ground and the ladder and the wall respectively, show that the least horizontal force which will move the lower extremity towards the wall is just greater than $\frac{W}{2}\left(\mu + \frac{1 + \mu}{1 - \mu'}\right)$ where W is the weight of the ladder.

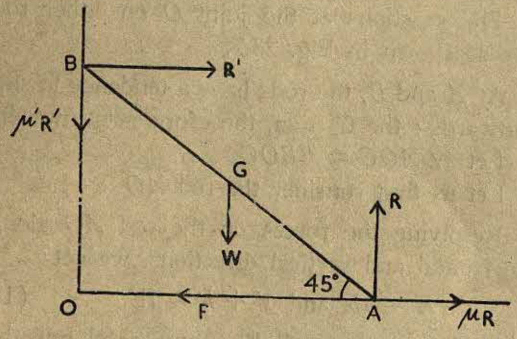


Fig. 23.8

Let AB be the uniform ladder of weight W and length $2l$, G being its midpoint. The ladder rests against the vertical wall OB . R,R' are the normal reactions, and μ, μ' the coefficients of friction at A and B respectively.

Let P be the horizontal force applied at A which makes A just on the point of motion towards the wall.

Since A is on the point of moving towards O, the frictional force μR at A acts in the direction OA. Again, A being on the point of motion towards O, B is on the point of motion outwards O ; hence the frictional force $\mu' R'$ at B acts in the direction BO. Also, the weight W acts vertically downwards through G.

Resolving the forces horizontally and vertically, we get

$$P = R' + \mu R \qquad (1)$$

$$\text{and} \quad R = W + \mu' R'. \qquad (2)$$

Taking moments about A, we have

$$R' \,.\, 2l \sin 45^\circ - \mu' R' \,.\, 2l \cos 45^\circ - W \,.\, l \cos 45^\circ = 0$$

or, $R'(1 - \mu') = \frac{1}{2}W$

$$\therefore \quad R' = \frac{W}{2(1 - \mu)} \qquad (3)$$

Eliminating R from (1) and (2), we get

$$P - R' = \mu W + \mu\mu' R'$$

$$\text{or,} \quad P = \mu W + R'(1 + \mu\mu')$$

$$= \mu W + \frac{W(1 + \mu\mu')}{2(1 - \mu')}$$

$$= \frac{W}{2} \cdot \frac{2\mu(1 - \mu') + 1 + \mu\mu'}{1 - \mu'}$$

$$= \frac{W}{2} \cdot \frac{2\mu - \mu\mu' + 1}{1 - \mu'}$$

$$= \frac{W}{2} \cdot \frac{1 + \mu + \mu(1 - \mu')}{1 - \mu'}$$

$$= \frac{W}{2}\left(\mu + \frac{1 + \mu}{1 - \mu'}\right).$$

$\therefore \quad P$ should be just greater than

$$\frac{W}{2}\left(\mu + \frac{1 + \mu}{1 - \mu'}\right)$$

to make the lower end of the rod move towards the wall.

Exercies 23.1

1. A body of weight 4 lbs. rests in limiting equilibrium on an inclined plane whose slope is 30°. Find the coefficient of friction and the normal reaction.
2. Determine the least force parallel to the plane which will pull a mass of 4 lbs. up a rough plane at an inclination of 30° to the horizon, if the angle of friction is 30°.

3. A body of weight 4 lbs. rests in limiting equilibrium on a rough plane whose slope is 30°. The plane being raised to a slope of 60°, find the force along the plane required to support the body.

4. Determine the least force parallel to the plane which will pull a mass of 20 lbs. down a rough plane at an inclination of 30° to the horizon, if the angle of friction is 60°.

5. A body of weight 40 lbs. rests on a rough horizontal plane whose coefficient of friction is 0.25 ; find the least force which acting horizontally would move the body.

Find also the least force which, acting at an angle of $\cos^{-1}\frac{3}{5}$ with the horizontal, would move the body.

6. A heavy uniform beam rests with one end on a rough horizontal plane, and the other on an equally rough plane inclined at 60° to the horizon. If the beam is on the point of motion down the plane when inclined at 30° to the horizon, show that the coefficient of friction is determined from the equation

$$\mu^2 + 4\mu\sqrt{3} - 3 = 0.$$

7. A uniform ladder is in equilibrium with one end resting on the ground and the other against a vertical wall. If the ground and the wall be both rough, their coefficients of friction being μ and μ' respectively, and if the ladder be on the point of slipping at both ends, show that the inclination of the ladder to the horizon is given by $\tan\theta = \dfrac{1 - \mu\mu'}{2\mu}$.

8. A heavy rod of length $2a$ lies over a rough peg with one extremity leaning against a rough wall, θ being its inclination to the wall. If c be the distance of the peg from the wall, and λ be the angle of friction both at the peg and at the wall, show that

(i) $\sin^2\theta\,\sin(\theta + 2\lambda) = \dfrac{c}{a}\cos^2\lambda$, when the rod is on the point of slipping downwards, and the point of contact of the rod and the wall lies below the peg ;

(ii) $\sin^2\theta\,\sin(\theta + 2\lambda) = \dfrac{c}{a}\cos^2\lambda$, when the rod is on the point of slipping upwards, and the point of contact of the rod with the wall lies below the peg ;

(iii) $\sin^3\theta = \dfrac{c}{a}\cos^2\lambda$, when the rod is on the point of sliding downwards and the point of contact of the rod with the wall lies above the peg.

9. A uniform ladder rests with one end on the rough horizontal ground and the other end against a rough vertical wall. The coefficients of friction at the lower and upper ends are 3/7 and 1/3 respectively. Determine the angle which the ladder makes with the ground when it is about to slip.

10. A ladder whose centre of gravity divides it into two portions of lengths a and b with one end on a rough horizontal floor and the other end against a rough vertical wall. If the coefficients of friction at the floor and the wall be μ and μ' respectively, show that the rod is inclined to the floor at an angle θ given by $\tan\theta = \dfrac{a - b\mu\mu'}{(a + b)\mu}$ when the equilibrium is limiting.

11. A ladder of length $2a$ is in contact with a vertical wall and a horizontal floor, the angle of friction being λ at each point of contact. If the weight of the ladder acts at a point distant Ka below the midpoint, prove that its limiting inclination θ to the vertical is given by $\cot\theta = \cot 2\lambda - K\operatorname{cosec} 2\lambda$.

12. A uniform ladder of weight w rests on a rough horizontal ground and against a smooth vertical wall, inclined at an angle α to the horizon. Prove that a man of weight W can climb to the top of the ladder without the

ladder slipping if $\frac{w}{W} > \frac{2(1 - \mu \tan \alpha)}{2\mu \tan \alpha - 1}$, μ being the coefficient of friction.

13. A uniform ladder rests in limiting equilibrium with one end on a rough horizontal plane and the other against a smooth vertical wall. A man then ascends the ladder. Show that, whatever his weight, he cannot go more than half-way up. What happens if the horizontal plane also be smooth ? Give reasons for your answer.

14. A uniform ladder of length $2a$ and weight w rests with one end against a rough vertical wall and the other upon an equally rough horizontal floor. Show that a man whose weight is W can never get nearer to the top of the ladder than $a \sin 2\lambda (w \cot 2\lambda + W \cot \lambda - (w + W) \tan \theta)/W$ where θ is the inclination of the rod to the horizon and λ is the angle of friction.

15. A particle is tied to one end of a light string, the other end of which is fixed to a point in a rough plane inclined to the horizon at an angle α which is greater than the angle of friction θ. Prove that the particle cannot remain at rest if the angle between the string and the line of greatest slope is greater than $\sin^{-1} (\tan \theta \cot \alpha)$.

16. A uniform ladder, of length l and weight W, rests with its foot on the rough ground, and its upper end against a smooth wall, the inclination to the vertical being α. A force P is applied horizontally to the ladder at a point distant c from the foot, so as to make the foot approach the wall. Prove that P must exceed $\frac{l\,W}{l - c} (\mu + \frac{1}{2} \tan \alpha)$ where μ is the coefficient of friction at the foot.

17. Prove that the least force, which applied to a uniform heavy sphere of weight W will maintain it in equilibrium against a rough vertical wall, is $W \cos \lambda$, provided λ, the angle of friction, is less than $\cos^{-1} \left(\frac{\sqrt{5} - 1}{2} \right)$.

18. Two equal weights connected by a string rest in equilibrium, one on each of two rough inclined planes of the same height and placed back to back, the string passing over a smooth pulley at the top of either plane. Show that the difference between the inclinations of the planes cannot exceed twice the angle of friction.

24 Machines

A **machine** is any contrivance (arrangement of bodies fitted together) to which a force can be conveniently applied at one point in order to raise a weight or overcome a resisting force acting at another point. The former force is called the **effort** or **power,** and the latter the **weight** or **resistance.** The **effort** is usually denoted by P and the **resistance** by either Q or W, the letter W being used when the resisting force is a heavy weight to be lifted by the machine.

In this chapter we shall discuss the working and properties of the following simple machines: (i) the lever, (ii) the inclined plane, (iii) the screw, (iv) the pulley and the system of pulleys.

In every case we suppose the machines to be devoid of friction and that all ropes and strings used in their working are light, inextensible and flexible.

24.1 Some definitions

24.1(*a*) *Mechanical advantage*

The mechanical advantage (or force-ratio) of a machine is a number which expresses the ratio of the resistance and the effort, i.e. **mechanical advantage** $= \dfrac{\textbf{Resistance}}{\textbf{Effort}} = \dfrac{\mathbf{Q}}{\mathbf{P}}$ **or** $\dfrac{\mathbf{W}}{\mathbf{P}}$.

The machines are constructed so that lesser effort is sufficient to overcome a greater resistance, i.e. the mechanical advantage is a ratio greater than unity. If the force-ratio be less than unity in any machine, it may be called a **mechanical disadvantage.**

REMARK

In a machine which gives mechanical advantage, the effort is less than the weight and therefore, the distance moved through by the effort is greater than the distance moved through by the resistance. Thus it would be appropriate to say that mechanical advantage is always gained at a proportionate diminution of the speed.

24.1(*b*) *Efficiency of a machine*

Efficiency of a machine is a number expressing the ratio of the work done **by** the machine to the work done **on** the machine; i.e.

$$\textbf{Efficiency} = \frac{\textbf{work done by machine}}{\textbf{work done on machine}}.$$

24.2 The Principle of work for machines

It is usually assumed that all machines work **without friction** and that their parts are **weightless.** With such machines, the following principle, called the **Principle of Work,** is universally true:

The work done by the machine against the resistance is equal to the work done on the machine by the effort.

REMARK

In any **actual machine,** there is always some friction to overcome; accordingly, the work done on the machine is always greater than the work done by the machine.

It, therefore, follows that the efficiency of a machine is always actually less than unity. For an **ideal machine,** the efficiency is unity.

COROLLARY

In the light of the Principle of Work (for ideal machines) enunciated above, the term **efficiency** may be redefined more appropriately.

If x and y be the distances moved by the effort P and the resistance Q (or W) respectively, then

$$\textbf{efficiency} = \frac{\mathbf{Wy}}{\mathbf{Px}} = \frac{\mathbf{W/P}}{\mathbf{x/y}}$$

$$= \frac{\textbf{Mechanical Advantage}}{\textbf{Velocity Ratio}}$$

24.3 The lever

A lever is a rigid rod which can turn freely about a fixed point, called the **fulcrum.** The effort is applied at any point of the lever so as to turn it about the fulcrum in order to raise a weight or overcome a resistance at any other point of the lever.

We are essentially concerned with the problem when a **lever** is **in equilibrium.** When it is in equilibrium, the algebraic sum of moments of the effort and the resistance about the fulcrum vanishes.

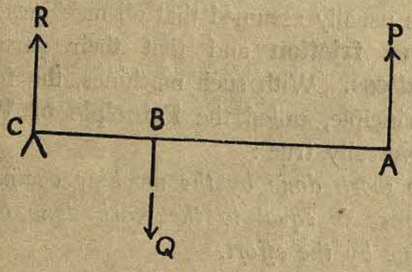


Fig. 24.1

Let C be the fulcrum of the lever AC. Let an effort P be applied at A perpendicular to AC to support a resistance Q applied at B perpendicular to BC.

By the condition of equilibrium,

algebraic sum of moments of P and Q about $C = 0$,

or, $P.AC - Q.BC = 0$

or, $\frac{Q}{P} = \frac{CA}{CB}$.

$\therefore$ Mechanical Advantage of the lever

$$= \frac{Q}{P} = \frac{\text{arm of effort}}{\text{arm of resistance}}$$

This shows that the effort and resistance are inversely proportional to the arms on which they act.

REMARK

The mechanical advantage of a lever may conveniently be made very great by applying the effort at the end of a long arm and the resistance very near the fulcrum.

COROLLARY

Let us suppose that the effort P and the resistance Q are not perpendicular to their arms CA and CB respectively. CM and CN are drawn perpendiculars to the lines of action of P and Q respectively.

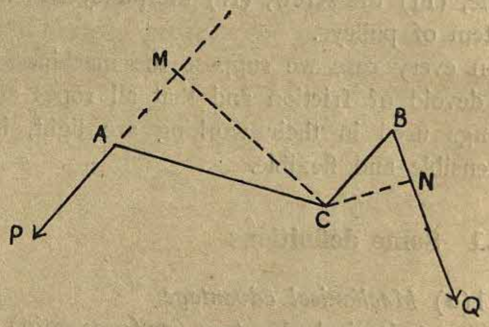


Fig. 24.2

Taking moments of P and Q about C, we have for equilibrium of the lever,

$$P.CM = Q.CN$$

∴ mechanical advantage of the lever

$$= \frac{Q}{P} = \frac{CM}{CN}$$

$$= \frac{\text{perpendicular distance of effort from fulcrum}}{\text{perpendicular distance of resistance from fulcrum}}.$$

24.3(*a*) *Classification of levers*

Levers are broadly divided into two categories : (1) **straight levers,** (2) **bent levers.**

When the arms of effort and resistance are in the same straight line, the lever is said to be straight ; otherwise, it is said to be a bent lever.

Straight levers are usually divided into three classes according to the relative positions of the points of application of the effort, resistance and the fulcrum.

In all our subsequent discussions on levers, we shall suppose the effort and the resistance to be parallel.

24.3(*b*) *A lever of the first class*

In a lever of the first class, the fulcrum is placed between the effort P and the resistance Q.

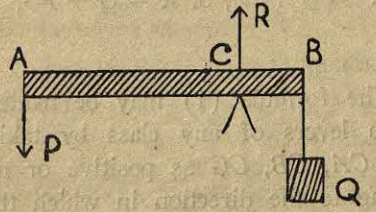


Fig. 24.3

Let R be the reaction at the fulcrum. Neglecting the weight of the lever, it is in equilibrium under the action of three forces P, Q, R.

∴ $R = P + Q$.

To lift a weight, the effort must be applied downwards so that the reaction at the fulcrum acts upwards, i.e. the lever presses downwards on the fulcrum.

Here mechanical advantage $= \frac{Q}{P} = \frac{AC}{BC}$,

and BC may either be greater or less than or equal to AC, in other words, the effort may either be greater or less than or equal to the weight to be lifted by the lever. Thus the mechanical advantage may be greater or less than or equal to unity.

REMARK

Examples of levers of the first class are a spade in digging ; a see-saw ; a poker used to raise the coals in the grāte ; the handle of a pump, etc. A pair of scissors, a pair of pincers are double levers of this class.

24.3(*c*) *A lever of the second class*

In a lever of the second class, the resistance Q is placed between the fulcrum and the effort P.

Here $Q = P + R$, $R = Q - P$.

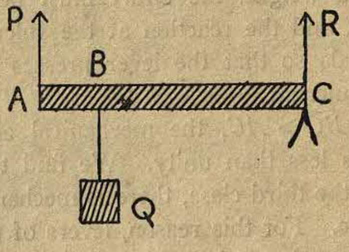


Fig. 24.4

To lift a weight, the effort must be applied upwards, and the reaction of the fulcrum also acts upwards so that the lever presses downwards on the fulcrum.

Since $AC > BC$, the mechanical advantage is always greater than unity.

REMARK

Examples of this class of levers are a wheelbarrow (the fulcrum being where the wheel touches the ground) ; a crowbar (with one end in contact with the ground) ; a cork squeezer ; an oar (assuming the end of the oar in contact with the water to be at rest). A pair of nutcrackers is a double lever of this class.

24.3(*d*) *A lever of the third class*

In a lever of the third class, the effort P is applied between the fulcrum and the resistance Q.

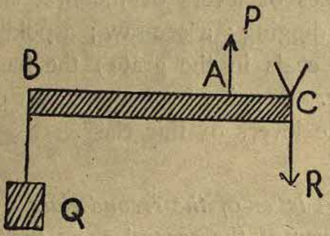


Fig. 24.5

Here $P = Q + R$, $R = P - Q$.
To lift a weight, the effort must be applied upwards, and the reaction at the fulcrum acts downwards so that the lever presses upwards on the fulcrum.

Since $BC > AC$, the mechanical advantage is always less than unity. We find that in a lever of the third class, there is mechanical disadvantage. For this reason, levers of the third class are rarely used.

REMARK

An example of a lever of the third class is the human forearm when it is used to support a weight placed on the palm of the hand, the fulcrum being the elbow and the tension exerted by the muscles being the effort. A pair of tongs is a double lever of this class.

24.3(*e*) *A heavy lever*

In our previous discussions, the weight of the lever was neglected. When the lever is heavy, i.e. it has a weight of its own, the condition of equilibrium may be found as in other cases by taking moments about the fulcrum.

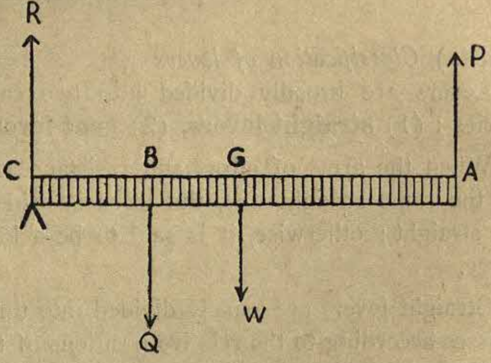


Fig. 24.6

Let the weight w of the lever act at its centre of gravity G (which may not necessarily be the midpoint). C is its fulcrum. An effort P is appiled at A to overcome a resistance Q at B.

Taking moments about C, the condition of equilibrium gives

$$P \,.\, CA = Q \,.\, CB + w \,.\, CG \qquad (1)$$

The reaction R at the fulcrum is given by

$$R + P = Q + w \text{ ; or } R = Q - P + w \qquad (2)$$

REMARKS

1. The formula (1) may be made applicable to levers of any class by taking the lengths CA, CB, CG as positive or negative according to the direction in which they are drawn from C.

2. All the formulae deduced above for levers of different classes are valid whether the lever is horizontal or inclined to the horizon, the only necessary condition being that the forces on the lever are all parallel and that their points of application are collinear with the fulcrum.

24.3 (*f*) *Illustrative examples*

EXAMPLE 1

If two forces applied to a weightless lever balance, and if the thrust on the fulcrum be ten times the difference of the forces, find the ratio of the arms.

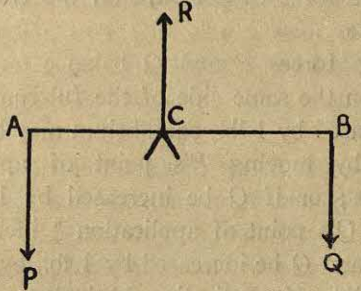


Fig. 24.7

Let C be the fulcrum, CA, CB the arms, and let a force P applied at A perpendicular to CA support a resistance Q applied at B perpendicular to CB. Let R be the reaction at the fulcrum.

We have for equilibrium,

$$R = P + Q \qquad (1)$$

and taking moments about C,

$$P \, . \, AC = Q \, . \, BC \; ; \text{ or, } \frac{AC}{BC} = \frac{Q}{P} \qquad (2)$$

By the given condition

$$R = 10(P - Q) \text{ assuming } P > Q \qquad (3)$$

By (1) & (3), $10(P - Q) = P + Q$;

$$\text{or, } 9P = 11Q \qquad (4)$$

By (2) & (4), $\dfrac{AC}{BC} = \dfrac{Q}{P} = \dfrac{9}{11}$.

Hence the arms are in the ratio 9 : 11.

EXAMPLE 2

A straight uniform lever is kept in equilibrium by weights of 12 and 5 lbs. respectively attached to the ends of its arms, and the length of one arm is double that of the other. What is the weight of the lever ?

The weight w of the lever AB acts at its midpoint G. Let C be the fulcrum and R the reaction there.

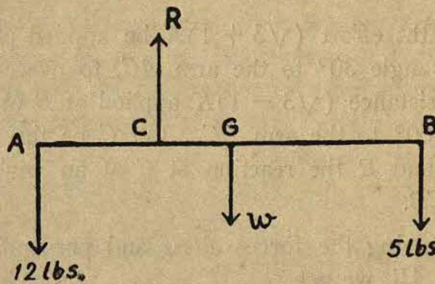


Fig. 24.8

Taking moments about C, the condition of equilibrium gives

$$12 \, . \, AC = 5 \, . \, BC + w \, . \, GC \qquad (1)$$

By the given condition,

$$BC = 2 \, AC.$$

Also $AG = BG = \frac{1}{2} AB$ and

$$AB = AC + BC = 3 \, AC.$$

$$\therefore \; GC = GA - CA = \tfrac{3}{2} AC - AC = \tfrac{1}{2} AC.$$

$\therefore$ (1) becomes $12 . AC = 10 . AC + \dfrac{w}{2} AC$

$$\text{or, } \frac{w}{2} = 12 - 10 = 2 \; ; \quad \therefore \; w = 4.$$

$\therefore$ The weight of the lever is 4 lbs.

EXAMPLE 3

A straight lever is acted on, at its extremities, by forces in the ratio $(\sqrt{3} + 1) : (\sqrt{3} - 1)$,

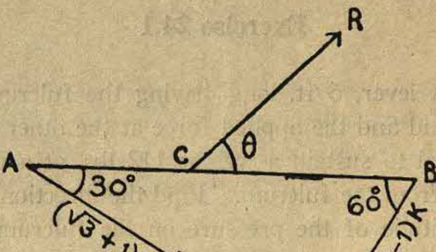


Fig. 24.9

and which are inclined at angles of 30° and 60° to its length. Find the magnitude of the thrust on the fulcrum and the direction in which it acts.

Let tht effort $(\sqrt{3}+1)K$ be applied at A at an angle 30° to the arm AC, to overcome the resistance $(\sqrt{3}-1)K$ applied at B at an angle 60° to the arm BC. Let C be the fulcrum and R the reaction at C at an angle θ with CB.

Resolving the forces along and perpendicular to AB, we get

$$\begin{aligned} X = R\cos\theta &= K(\sqrt{3}+1)\cos 30^\circ \\ &\qquad - K(\sqrt{3}-1)\cos 60^\circ \\ &= \frac{K(\sqrt{3}+1)\sqrt{3}}{2} - \frac{K(\sqrt{3}-1)}{2} \\ &= 2K \quad (1) \end{aligned}$$

$$\begin{aligned} Y = R\sin\theta &= K(\sqrt{3}+1)\sin 30^\circ \\ &\qquad + K(\sqrt{3}-1)\sin 60^\circ \\ &= \frac{K(\sqrt{3}+1)}{2} + \frac{K(\sqrt{3}-1)\sqrt{3}}{2} \\ &= 2K \quad (2) \end{aligned}$$

Squaring and adding (1) and (2), we get

$$R = \sqrt{X^2+Y^2} = \sqrt{8K^2} = 2\sqrt{2}K.$$

Also, $\tan\theta = \frac{Y}{X} = 1 = \tan 45^\circ$,

$\therefore \quad \theta = 45^\circ.$

Taking $K = 1$, we find that the thrust on the fulcrum is $2\sqrt{2}$ at an angle of 45° with the lever.

Exercise 24.1

1. A lever, 6 ft. long, having the fulcrum at one end and the applied force at the other end, is used to sustain a weight 112 lbs. at a point 2 ft. from the fulcrum. Find the direction and magnitude of the pressure on the fulcrum.

2. Where must the fulcrum be so that a weight of 6 lbs. may balance a weight of 8 lbs. on a straight weightless lever 7 feet long ?

If each weight be increased by 1 lb., in what direction will the lever turn ?

3. Two weights P and Q balance on a weightless lever, the fulcrum being $1\frac{1}{4}$ inches from the middle point of the lever. If each weight be increased by 1 lb., the fulcrum must be moved $\frac{1}{4}$ inch in order to maintain equilibrium. Find the force of pressure on the fulcrum in each case.

4. Two forces P and Q balance on a lever acting on the same side of the fulcrum. If P be increased by 1 lb., equilibrium may be maintained by moving P's point of application 2 inches ; or if Q be increased by 1 lb., by moving Q's point of application $\frac{3}{4}$ inch ; or if both P and Q be increased by 1 lb., by moving both points of application $1\frac{1}{3}$ inches. Find P and Q.

5. A straight weightless lever has for its fulcrum a hinge at one end A, and from a point B is hung a body of weight W. If the strain at the hinge must not exceed $\frac{1}{2}W$ in either direction, upwards or downwards, show that the effort must act somewhere within a space equal to $4/3\,AB$.

6. A straight horizontal lever has for its fulcrum a hinge at one end A, and at a point B is hung the weight W. If the strain on the hinge must not exceed $\frac{1}{3}W$ either upwards or downwards, prove that the power must act somewhere within a space equal to $\frac{3}{4}AB$.

7. A straight lever AB, 12 feet long, balances about a point 1 ft. from A when a weight of 13 lbs. is suspended from A. It will balance about a point which is 1 ft. from B when a weight of 11 lbs. is suspended from B. Show that the centre of gravity of the lever is 5 inches from the middle point of the lever.

8. Two weights of 4 lb. and 8 lb. balance when suspended from the ends of a straight weightless lever with the fulcrum 1 ft. from the larger weight. When P lb. are added to each weight, the fulcrum has to be shifted a

distance of 2 inches. Find the value of P and the length of the lever.

9. A straight uniform lever of length 5 ft. and weight 10 lbs. has its fulcrum at one end and weights of 3 and 6 lbs. are fastened to it at distances of 1 and 3 ft. respectively from the fulcrum; it is kept horizontal by a force at its other end; find the thrust on the fulcrum.

10. Explain carefully why a man stands on the bottom rung of a ladder, and holds on to another rung as low as he can, when another man is lifting the ladder.

11. Why cannot a man, sitting in a basket, lift himself and the basket off the ground by pulling at the handles of the basket ?

12. In any lever, find the mechanical advantage when the effort is applied at (i) the point of application of the resistance, (ii) the fulcrum.

13. A straight lever whose arms are a and b ft. respectively, is in equilibrium under forces P and Q at its extremities, whose lines of action make, with the lever, angles α and β respectively. Find the ratio of P to Q, and the pressure on the fulcrum.

14. Show that in a weightless lever of the first class the power, the weight and the reaction of the fulcrum form two unlike couples of equal moments when the lever is in equilibrium.

24.4 The inclined plane

By means of an inclined plane, a heavy body can easily be raised through a great distance, the effort required being less than the weight raised. The principle and use of a rough inclined plane has already been discussed in Chapter 23. We shall consider here the equilibrium of a heavy body on a smooth inclined plane under any force. It is assumed that all forces on the body lie in a vertical plane through the line of greatest slope of the inclined plane. The plane being smooth, the only reaction between it and any body resting on it is perpendicular to the inclined plane.

24.4 (*a*) *Equilibrium of a heavy body on a smooth inclined plane under any effort*

Let a heavy body of weight W be placed at D on a smooth inclined plane AC of inclination α to the horizon. Let P be the effort applied on the body and R the reaction of the plane.

Let AB be the horizontal line through A, DE the perpendicular on AB and DF perpendicular to the inclined plane as shown in Figs. 24.10, 24.11 and 24.12.

In each of these figures,

$$\angle EDF = 90^\circ - \angle ADE = \angle DAE = \alpha.$$

We now consider the following cases.

CASE I : *Let the effort P act up the plane parallel to the line of greatest slope.*

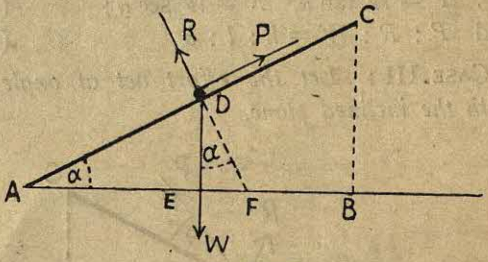


Fig. 24.10

Since the three forces P, R, W acting at D are in equilibrium, we have by applying Lami's Theorem,

$$\frac{P}{\sin(R, W)} = \frac{R}{\sin(W, P)} = \frac{W}{\sin(P, R)}$$

or, $$\frac{P}{\sin(180^\circ - \alpha)} = \frac{R}{\sin(90^\circ + \alpha)} = \frac{W}{\sin 90^\circ}$$

or, $$\frac{P}{\sin\alpha} = \frac{R}{\cos\alpha} = W. \qquad (1)$$

$$\therefore \; P = W\sin\alpha, \quad R = W\cos\alpha \qquad (2)$$

The lines AC, AB, BC are often called the *length*, *base* and *height* respectively of the plane, and are represented by l, b, h respectively.

From (1), $$\frac{P}{h/l} = \frac{R}{b/l} = \frac{W}{l/l}$$

so that $$P : R : W = h : b : l \qquad (3)$$

CASE II : *Let the effort act horizontally.*

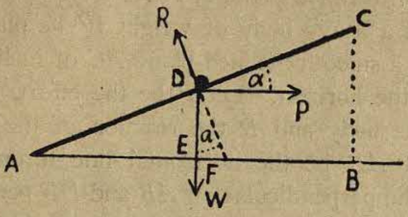


Fig. 24.11

As in Case I, we have by Lami's Theorem

$$\frac{P}{\sin(180^\circ - \alpha)} = \frac{R}{\sin 90^\circ} = \frac{W}{\sin(90^\circ + \alpha)}$$

$$\text{or, } \frac{P}{\sin\alpha} = \frac{W}{\cos\alpha} = R \qquad (4)$$

$$\therefore \; P = W\tan\alpha, \quad R = W\sec\alpha\,; \qquad (5)$$

$$\text{and } \; P : R : W = h : l : b \qquad (6)$$

CASE III : *Let the effort act at angle θ with the inclined plane.*

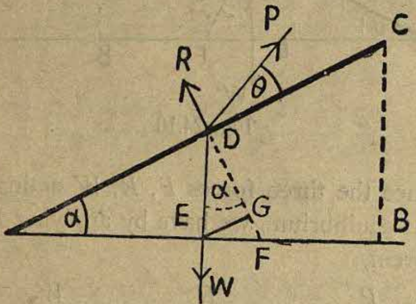


Fig. 24.12

As in the previous two cases, we have by Lami's Theorem,

$$\frac{P}{\sin(180^\circ - \alpha)} = \frac{R}{\sin(90^\circ + \theta + \alpha)} = \frac{W}{\sin(90^\circ - \theta)},$$

$$\text{or, } \frac{P}{\sin\alpha} = \frac{R}{\cos(\theta + \alpha)} = \frac{W}{\cos\theta} \qquad (7)$$

$$\therefore \; P = W\frac{\sin\alpha}{\cos\theta}, \; R = W\frac{\cos(\theta + \alpha)}{\cos\theta} \qquad (8)$$

REMARKS

1. Let us draw EG parallel to P to meet DF in G.

DEG is a triangle of forces P, R, W.

$\therefore \; P : R : W = EG : GD : DE.$

2. Putting $\theta = 0$ in Case III, we get Case I.

Putting $\theta = -\alpha$ in Case III, we get Case II.

3. Taking up Case III which is a more general one, we may easily verify the Principle of Work.

Let d be the distance through which the body moves along the plane. Then $d\cos\theta$ is the distance through which the point of application of P moves along its direction, and $d\sin\alpha$ is the vertical distance through which the weight moves.

$\therefore$ Work done by the effort $P = Pd\cos\theta$

$= Wd\sin\alpha$ by (8)

$=$ Work done against the weight.

24.4 (*b*) *Illustrative examples*

EXAMPLE 1

A weight of 9 lbs. is pulled up an inclined plane of which the height is $1\frac{1}{2}$ ft. and the base is 2 ft. What force (i) acting horizontally, (ii) acting along the plane, is required for the purpose.

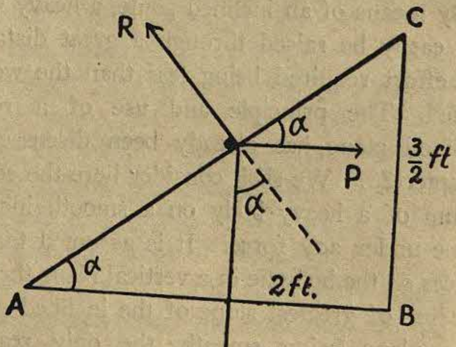


Fig. 24.13

Let $AB = 2$ ft., $BC = \frac{3}{2}$ ft. be respectively the base and height of the plane AC inclined to the horizon at an angle α. R is the normal reaction of the plane and P is the force applied to the body.

(i) Let P be applied horizontally.

Resolving along the plane, the condition of equilibrium gives

$$P \cos \alpha = 9 \cos (90^\circ - \alpha) = 9 \sin \alpha$$

$$\therefore \quad P = 9 \tan \alpha = 9 \cdot \frac{BC}{AB} = 9 \cdot \frac{3}{4}$$

$$= \frac{27}{4} = 6\frac{3}{4}$$

$\therefore$ P must be slightly greater than $6\frac{3}{4}$ lbs. so that the weight may be pulled up the plane.

(ii) When P acts up the plane along the line of greatest slope, we have by resolving the forces along the plane, $P = 9 \sin \alpha$.

$$\text{Now } AC = \sqrt{AB^2 + BC^2}$$

$$= \sqrt{4 + \frac{9}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}.$$

$$\therefore \quad \sin \alpha = \frac{BC}{AC} = \frac{3/2}{5/2} = \frac{3}{5}.$$

$$\therefore \quad P = 9 \cdot \frac{3}{5} = \frac{27}{5} = 5\frac{2}{5}$$

$\therefore$ P must be slightly greater than $5\frac{2}{5}$ lbs.

EXAMPLE 2

A body of weight W is supported on a smooth inclined plane, by a force P inclined to the plane, and the pressure on the plane is R.

Given that $P = \frac{1}{\sqrt{3}} W$, $R = \frac{1}{\sqrt{3}} W$,

find the inclination of the plane, and the direction of P.

Let the plane be inclined to the horizon at an angle α, and let the effort P act on the body at an angle θ with the line of greatest slope of the inclined plane [see Fig. 24.12].

Resolving along and perpendicular to the plane, we have

$$P \cos \theta = W \sin \alpha \qquad (1)$$

$$\text{and} \quad R + P \sin \theta = W \cos \alpha \qquad (2)$$

$$\therefore \quad R = W \cos \alpha - P \sin \theta$$

$$= W \cos \alpha - \frac{W \sin \alpha}{\cos \theta} \sin \theta$$

$$= \frac{W(\cos \alpha \cos \theta - \sin \alpha \sin \theta)}{\cos \theta}$$

$$= \frac{W \cos (\alpha + \theta)}{\cos \theta} \qquad (3)$$

$$\therefore \quad \frac{1}{\sqrt{3}} W = \frac{W \sin \alpha}{\cos \theta}; \text{ or, } \frac{\sin \alpha}{\cos \theta} = \frac{1}{\sqrt{3}};$$

$$\text{and} \quad \frac{1}{\sqrt{3}} W = \frac{W \cos (\alpha + \theta)}{\cos \theta};$$

$$\text{or, } \frac{\cos (\alpha + \theta)}{\cos \theta} = \frac{1}{\sqrt{3}}.$$

$$\therefore \quad \frac{\sin \alpha}{\cos \theta} = \frac{\cos (\alpha + \theta)}{\cos \theta}$$

$$\text{or, } \sin \alpha = \cos (\alpha + \theta) = \sin \{90^\circ - (\alpha + \theta)\}$$

$$\text{or, } \quad \alpha = 90^\circ - (\alpha + \theta)$$

$$\text{or, } \quad 2\alpha = 90^\circ - \theta; \quad \therefore \quad \alpha = 45^\circ - \frac{\theta}{2}$$

$$\therefore \quad \frac{1}{\sqrt{3}} = \frac{\sin \alpha}{\cos \theta} = \frac{\sin\left(45^\circ - \frac{\theta}{2}\right)}{\cos \theta}$$

$$= \frac{\frac{1}{\sqrt{2}}\left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2}\right)}{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}}$$

$$\text{or, } \quad \sqrt{\frac{2}{3}} = \frac{1}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}}$$

$$\text{or, } \quad \cos \frac{\theta}{2} + \sin \frac{\theta}{2} = \sqrt{\frac{3}{2}}$$

$$\text{Squaring,} \quad 1 + \sin \theta = \frac{3}{2}$$

$$\text{so that} \quad \sin \theta = \tfrac{3}{2} - 1 = \tfrac{1}{2} = \sin 30^\circ,$$

$$\therefore \quad \theta = 30^\circ.$$

$$\therefore \quad \frac{1}{\sqrt{3}} = \frac{\sin \alpha}{\cos \theta} = \frac{\sin \alpha}{\cos 30^\circ}$$

$$= \frac{\sin \alpha}{\frac{\sqrt{3}}{2}} = \frac{2 \sin \alpha}{\sqrt{3}}$$

$$\text{or, } \quad \sin \alpha = \tfrac{1}{2} = \sin 30^\circ, \quad \therefore \quad \alpha = 30^\circ.$$

∴ Inclination of the plane to the horizon is 30° and the effort P acts at an angle 30° with the plane.

Exercise 24.2

1. Find (i) the horizontal force, and (ii) the force up the plane required to support a weight of 85 kg. on an incline of 8 in 17 of length 34 metres.

2. What force (i) acting horizontally, and (ii) acting up the plane could keep a mass of 10 kg. at rest on an incline of 60° ? Find also the reaction in each case.

3. A body rests on an inclined plane, being supported by a force acting up the plane equal to half its weight. Find the inclination of the plane to the horizon and the reaction of the plane.

4. A body of weight 10 lb. rests on a smooth plane inclined at an angle of 30° to the horizon. Find the least force required to sustain it and the reaction of the plane.

5. A body rests on a plane, inclined at an angle of 60° to the horizon, and is supported by a force inclined at an angle of 30° to the horizon. Show that the force and the reaction of the plane are each equal to the weight of the body.

6. A body rests on a plane inclined to the horizon at an angle α. If the reaction of the plane be equal to the effort applied, show that the inclination of the effort to the inclined plane is $(90° - 2\alpha)$.

7. A weight $2W$ is kept in equilibrium on an inclined plane by a horizontal force W, and a force W acting parallel to the plane. Find the ratio of the length, base, and height of the plane, and also the pressure on the plane.

8. A body of 5 lbs. wt. is placed on a smooth plane inclined at 30° to the horizon, and is acted on by two forces, one equal to the weight of 2 lbs. and acting parallel to the plane and upwards, and the other equal to P and acting at an angle of 30° with the plane. Find P and the reaction of the plane.

9. Two weights support each other on two smooth inclined planes of the same height, the weights being connected by a string passing over a small pulley at the junction of the planes. The angles of the planes are 30° and 60°. What is the ratio of the weights and the tension of the string ?

10. On two inclined planes, of equal height, two weights are respectively supported, by means of a string passing over the common vertex and parallel to the planes. The length of one plane is double its height, and the length of the other plane is double its base. Show that the reaction of one plane is three times the reaction of the other.

11. AB and AC are two smooth planes inclined to the horizon at angles α and β respectively, and meeting at their highest point A. Two weights W and V, one on each slope, balance one another when connected by a string passing over a smooth pulley at A. If, when the inclination of each plane is doubled, equilibrium is maintained by replacing V by V', prove that $V \cos \alpha = V' \cos \beta$.

12. If a force P, acting parallel to an inclined plane and supporting a mass of weight W, produces on the plane a thrust R, show that the same power, acting horizontally and supporting a mass of weight R, will produce on the plane a thrust W.

13. A weight is supported on a smooth inclined plane by a string attached to the top of the plane. At three different inclinations of the plane, the tensions are proportional to $\sqrt{3}$, $\sqrt{2}$, 1 respectively. Find the three inclinations given that they are in arithmetical progression.

14. A weight is supported on a smooth plane of inclination α, by a string inclined to the horizon at an angle γ. If the slope of the

plane be increased to β, and the slope of the string unaltered, the tension of the string is doubled. Prove that $\cot\alpha - 2\cot\beta = \tan\gamma$.

15. A body of weight W is supported on a smooth inclined plane, by a force P inclined to the plane, and the pressure on the plane is R.

Given that

$$P = \tfrac{1}{2}\sqrt{2}\,W,\ R = \tfrac{1}{2}(\sqrt{3} - 1)W.$$

Find the inclination of the plane and the direction of P.

16. Find the pressure on a smooth plane inclined at an angle α to the horizon, when a weight W is supported on it by a force whose line of action makes an angle β with the horizon.

If the pressure on the plane be in arithmetic mean between the weight and the force, and if $\beta = 2\alpha$, find $\sin\beta$.

24.5 The screw

The **screw** consists of a **bolt** of circular cross-section, with its axis OM. The surface of the bolt carries a **thread** or has a **groove** cut in it along a spiral curve. The screw works in a **collar** or **nut** C through which a hole is bored. The thread on the surface of the bolt is equally inclined at all points to the plane perpendicular to the axis of the bolt.

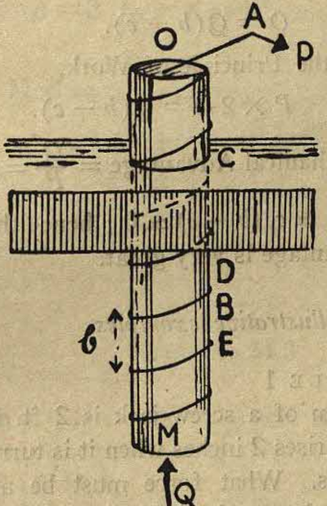


Fig. 24.14

As the screw turns in a fixed collar, it moves forward parallel to its axis. In each complete revolution, the distance moved forward is equal to the distance between two consecutive threads, measured parallel to the axis. This distance is known as the **step.**

The **effort** applied to the screw tends to rotate the bolt, and has a **moment** about OM in a plane perpendicular to OM. The effort is a single force P applied at the end of a long arm OA projecting at right angles to OM.

In a common screw-press or screw-jack, two equal and opposite forces constituting a **couple** are applied at right angles to the extremities A, B of the arm which projects in both directions [see Fig. 24.15].

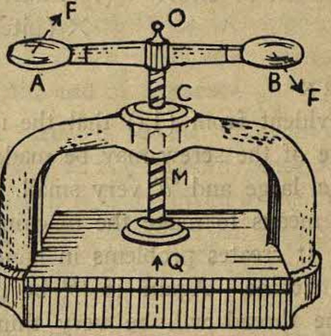


Fig. 24.15

Let a be the length of the arm OA (Fig. 24.14) and let the effort P be applied perpendicularly at A. Let P overcome a resistance Q acting along the axis OM. It is further assumed that the screw is smooth and b is its step. In one complete revolution of the screw, A (the point of application of the effort P) describes a circle of radius a with centre O. Therefore, the point A and hence the effort P

moves through a distance $2\pi a$, while the screw moves forward a distance b.

∴ By the Principle of Work,

work done by P = work done against Q

or, $P \times 2\pi a = Q \times b$.

∴ *Mechanical Advantage of the screw*

$$= \frac{Q}{P} = \frac{2\pi a}{b} \qquad (1a)$$

$$= \frac{\text{circumference of the circle with the arm as radius}}{\text{step of the screw}}. \qquad (1b)$$

The quantity $b/2\pi$ is called the **pitch** of the screw.

We find that

$$\text{moment of } P \text{ about } OM = P\,.\,a = Q \times b/2\pi\,;$$

or, moment of effort = (resistance) × (pitch) (2)

REMARK

It is evident from (1a) that the mechanical advantage of the screw may be made large by making 'a' large and 'b' very small. To make 'a' large means to make the machine huge in its size. It creates problems in handling such a huge machine. Again, if 'b' be made very small, the thread will be very thin and the machine will be very weak. It is, therefore, necessary to modify the construction of the machine without making 'a' large or 'b' small.

This is done in the **differential screw** which will be discussed next.

24.5(*a*) *The differential screw*

In this machine a hollow screw M, which is turned by the lever, works in a collar C. Within the hollow screw M, another screw N of smaller diameter works. The screw N is attached to the top of the press A and works in a collar which is cut in the lower extremity of the screw M. The screw N cannot rotate but can only move in the direction of its length.

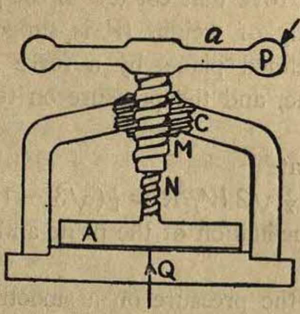


Fig. 24.16

The step b of the screw M is slightly greater than the step c of the screw N. Let the effort P be applied at the end of the lever of length a to overcome the resistance Q of the press.

For one complete revolution of the lever, work done by the effort $P = P \times 2\pi a$.

Again, in one complete revolution the screw M travels downwards through a distance b while the screw N penetrates a distance c into the screw M. Thus the press A is forced down through a distance $(b - c)$ only.

∴ Work done against the resistance

$$Q = Q(b - c).$$

∴ By the Principle of Work,

$$P \times 2\pi a = Q(b - c).$$

$$\therefore \text{ Mechanical Advantage} = \frac{Q}{P} = \frac{2\pi a}{b - c} \quad (1)$$

Since $b \simeq c$, $(b - c) \simeq 0$; hence the mechanical advantage is very great.

24.5(*b*) *Illustrative examples*

EXAMPLE 1

The arm of a screw-jack is 2 ft. long, and the screw rises 2 inches when it is turned round nine times. What force must be applied to produce a thrust of a half ton weight ? (1 ton = 2240 lbs.)

Here the arm $a = 2$ ft., $Q = 1120$ lbs.,

$$\text{step } b = \frac{\cancel{2}}{9\times\underset{6}{\cancel{12}}}\text{ ft.} = \frac{1}{54}\text{ ft.}$$

We know that $\frac{Q}{P} = \frac{2\pi a}{b}$

so that

$$P = \frac{bQ}{2\pi a} = \frac{1\times \overset{\overset{\overset{\overset{70}{\cancel{140}}}{\cancel{280}}}{\cancel{560}}}{\cancel{1120}}\times 7}{\underset{27}{\cancel{54}}\times\cancel{2}\times\underset{11}{\cancel{22}}\times\cancel{2}} = \frac{490}{297}$$

$\therefore$ The required force $= \frac{490}{297}$ lbs. wt.

EXAMPLE 2

Find the mechanical advantage in a differential screw having 5 threads to the inch and 6 threads to the inch, the effort being applied at the circumference of a wheel of diameter 4 ft.

For the differential screw,

the mechanical advantage $= \frac{2\pi a}{b - c}$.

Here $a = 2$ ft., $b = \frac{1}{5}$ inch $= \frac{1}{60}$ ft.,

$c = \frac{1}{6}$ inch $= \frac{1}{72}$ ft.

$\therefore$ Mechanical Advantage

$$= \frac{2\times 22\times 2}{7\times\left(\frac{1}{60} - \frac{1}{72}\right)} = \frac{2\times 22\times 2\times 360}{7}$$

$$= \frac{31680}{7} = 4525\tfrac{5}{7}.$$

Exercise 24.3

1. In a screw-press a power equal to 16 lb. weight, acting on an arm 3 ft. long, produces a thrust of half a ton. What is (i) the step, (ii) the pitch of the screw ?
2. The arm of a screw-jack is 1 yard long and the screw has two threads to the inch. What force must be applied to the arm to raise 1 ton ?
3. If the effort be applied at the end of an arm of length 1 ft. and if the screw makes seven complete turns in 1 ft. of its length, find the effort that will support a weight of 1 ton.
4. The arm of a screw is 2 ft. long and it has 10 threads per foot of its length. What thrust will the screw produce if the effort be a force of 112 lbs. wt ?
5. If a power of 10 lbs. wt. acting on an arm 2 ft. long produces in the screw press a thrust of one ton weight, what is the step of the screw ?
6. If the lever by which a screw in worked be 6 ft. in length, determine the distance between two successive threads of the screw, in order that a thrust of 10 lbs. wt. applied to each end of the lever may produce a thrust of 1000 lbs. wt. at the end of the screw.
7. A screw is formed upon a cylinder whose length is 1 ft. and circumference 3 inches. How many turns must be given to the thread in order that an effort of 4 lbs. wt., applied tangentially at the edge of the cylinder, may support a weight of 2 cwt. ?
8. Find the mechanical advantage in a differential screw, the larger screw having 8 threads to the inch and the smaller 9 threads, the length of the effort-arm being 1 ft.

24.6 The pulley

A **pulley** consists of a wheel of wood or metal with a **groove** cut round its circumference. A string or rope or chain passes round the pulley through the groove. It turns on an axis or **axle,** which passes through its centre and is perpendicular to its plane. The axle is fixed in a framework called the **block** or **sheave.** This block is either **fixed** or **movable.**

24.6(*a*) *Single fixed pulley*

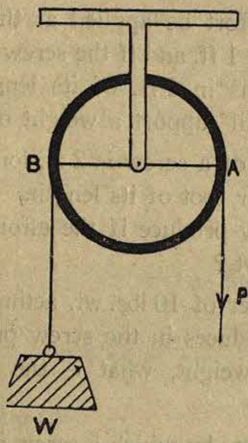


Fig. 24.17

The block of the pulley is fixed here. The weight W is attached to one end of the string passing round the groove while the effort P is applied at the other end. The effort P may be vertical [Fig. 24.17] or inclined [Fig. 24.18].

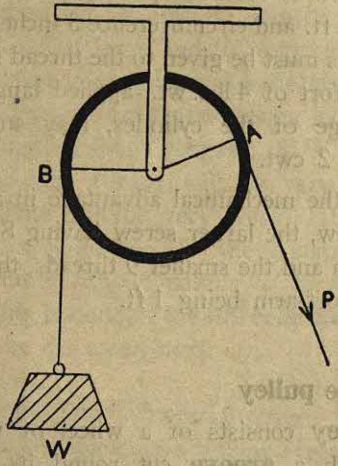


Fig. 24.18

Let us assume that the string is smooth. Since the wheel is supported by the axle only, the condition of equilibrium gives,

moment of P about O = moment of W about O,

or, $P \times OA = W \times OB$

$\therefore\ P = W$ [$\because\ OA = OB$ = radius of the pulley].

$$\therefore \quad \text{Mechanical Advantage} = \frac{W}{P} = 1.$$

REMARKS

1. Since $P = W$, the tension is uniform throughout the string.

2. Since $P = W$, the force to be applied to lift the weight W is the same as it would be without the pulley. The only convenience in using the pulley is that the force can be applied in any convenient direction.

3. When P is vertical, pressure on the fixed block $= P + W = 2W$.

When P is not vertical, let 2θ be the angle between P and W. In this case, pressure on the block $= 2W \cos \theta$.

24.6(*b*) *Single movable pulley*

Here the weight W is attached to the block. The effort P is applied at one end of the string passing round the pulley while the other end of the string is fixed.

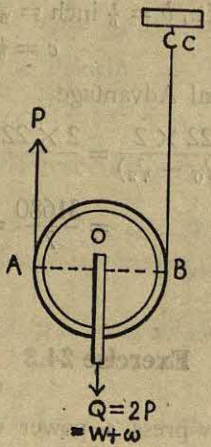


Fig. 24.19

If w be the weight of the pulley, the total weight to be raised by the effort P is

$$Q = W + w.$$

The tension in each part of the string is P. Since the tensions at both A and B support the total weight Q, we must have

$$Q = P + P = 2P.$$

$$\therefore \quad \text{Mechanical Advantage} = \frac{Q}{P} = 2.$$

Also $W + w = 2P$.

REMARK

This system of pulleys is frequently used in cranes.

24.6(*c*) *The single-string system* (*Second System*) *of pulleys*

Here the pulleys are arranged in two blocks —the upper block is fixed to a support while the lower block is attached to the weight to be lifted.

The number of pulleys in the two blocks may be equal or unequal. In Fig. 24.20, the number of the pulleys in each of the blocks is the same and the string is tied to the upper block (which is fixed). In Fig. 24.21, the number of pulleys in the two blocks is not the same; the upper block has one pulley more than the lower block and the string is tied to the lower block. A single string passes round all the pulleys.

Let there be n portions of the string at the lower block, corresponding to the n number of pulleys in the system. Let the effort P lift a weight W, hanging from the movable block of weight w, through a height h. Then each of the n portions of the string becomes shortened in length by an amount h. This means that P pulls the end of the string down by a distance nh.

$\therefore$ By the Principle of Work,

$$Wh + wh = P \,.\, nh$$

$$\text{or,} \quad W + w = nP. \tag{1}$$

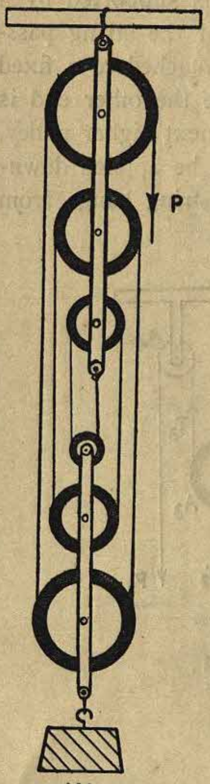


Fig. 24.20

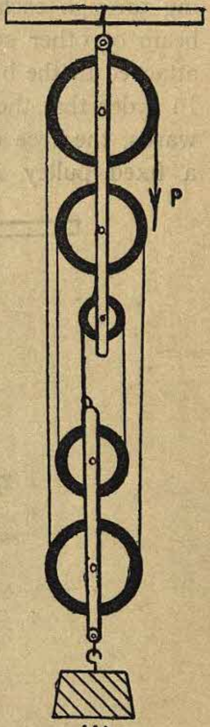


Fig. 24.21

$\therefore$ Mechanical Advantage

$$= \frac{W + w}{P} = n. \tag{2}$$

REMARKS

1. If the weights of the pulleys are neglected, we have $W = nP$ so that mechanical advantage $= \dfrac{W}{P} = n$.

2. In order that the effort may be applied downwards, the free end of the string must hang from a fixed pulley.

24.6(*d*) *The separate-string system* (*First System*) *of pulleys*

In this system, the pulleys are taken to be movable and each of them is supported by a separate string. One end of the string passing through each pulley is attached to a fixed beam or other support while the other end is attached to the block of the next higher pulley. In order that the effort may be applied downwards, the free end of the string hangs from a fixed pulley A_4.

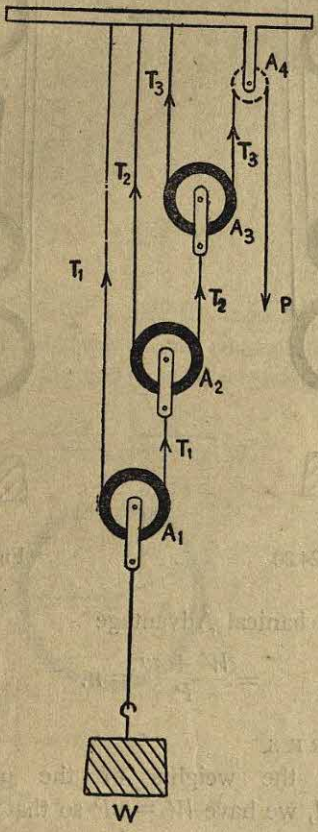


Fig. 24.22

Let there be n pulleys A_1, A_2, ... having weights w_1, w_2, ... respectively.

Let the free end of the string be pulled with a force P and let the system support a weight W attached to the block of the lowest pulley A_1. If the effort P lifts the weight W through a height h, then the lowest pulley A_1 will rise through a height h; the one above it, A_2, will rise through $2h$; the next A_3 through $4h$, and so on.

Hence P will move its point of application down through a distance $2^n h$.

∴ By the Principle of Work, we have

$$P \times 2^n h = Wh + w_1 h + 2w_2 h + 4w_3 h + \cdots + 2^{n-1} w_n h;$$

$$\text{or,}\quad 2^n P = W + w_1 + 2w_2 + 4w_3 + \cdots + 2^{n-1} w_n;$$

$$\text{or,}\quad P = \frac{W}{2^n} + \frac{w_1}{2^n} + \frac{w_2}{2^{n-1}} + \frac{w_3}{2^{n-2}} + \cdots + \frac{w_n}{2}. \tag{1}$$

REMARKS

1. If the pulleys be of equal weight w, then

$$2^n P = W + (1 + 2 + 2^2 + \cdots + 2^{n-1}) w$$
$$= W + (2^n - 1) w.$$

2. If we leave out of account the weights of the pulleys themselves, we have

$$2^n P = W \quad [\because w_1 = w_2 = \cdots = w_n = 0].$$

$$\therefore \quad \text{Mechanical Advantage} = \frac{W}{P} = 2^n.$$

It is evident that the mechanical advantage increases with increasing n.

3. When the weights of the pulleys are taken into account, the greater the weights of the pulleys, the greater is P. Hence the effort P can be made small only when the pulleys are taken as light as possible.

24.6(*e*) *The inverted separate-string system (Third System) of pulleys*

In this system, the pulleys are movable except the uppermost one. Each of the pulleys is supported by a separate string. One end of the string is attached to a rod from which the weight W is suspended while the other end is attached to the next lower pulley. The effort P is applied at the free end of the string which passes over the lowest pulley A_1

[Fig. 24.23]. We assume the portions of the string to be parallel and also, that there is no friction.

Let there be n pulleys of which the upper one is fixed and $(n-1)$ movable. Let the effort P lift the weight W.

Let $A_1, A_2, A_3 \dots$ be the movable pulleys, beginning from the lowest, and let $T_1, T_2, T_3 \dots$ be the tensions of the strings passing round these pulleys.

Since A_1 is smooth, we have $T_1 = P$.

CASE I : *Let the weights of the pulleys be neglected.*

Since $A_1, A_2, A_3, \dots$ are in equilibrium, we have

$$\begin{aligned} T_2 &= 2T_1 = 2P, \\ T_3 &= 2T_2 = 2^2P, \\ T_4 &= 2T_3 = 2^3P \\ &\dots\dots\dots\dots \\ &\dots\dots\dots\dots \\ T_{n-1} &= 2T_{n-2} = 2^{n-2}P, \\ T_n &= 2T_{n-1} = 2^{n-1}P. \end{aligned}$$

Since the bar from which the weight W is suspended, is in equilibrium, we have

$$\begin{aligned} W &= T_1 + T_2 + \dots + T_n \\ &= (1 + 2 + 2^2 + 2^3 + \dots + 2^{n-1})P \\ &= (2^n - 1)P \qquad (1) \end{aligned}$$

$\therefore$ Mechanical Advantage

$$= \frac{W}{P} = 2^n - 1 \qquad (2)$$

CASE : II *Let the weights of the pulleys be taken into account.*

Let the weights of the movable pulleys $A_1, A_2, \dots\dots, A_{n-1}$ be $w_1, w_2, \dots, w_{n-1}$ respectively

We first consider the equilibrium of the four pulleys shown in Fig. 24.23.

Considering the equilibrium of the pulleys taken in order, we find

$$\begin{aligned} T_2 &= 2T_1 + w_1 = 2P + w_1, \\ T_3 &= 2T_2 + w_2 = 2^2P + 2w_1 + w_2 \\ T_4 &= 2T_3 + w_3 = 2^3P + 2^2w_1 \\ &\qquad + 2w_2 + w_3. \end{aligned}$$

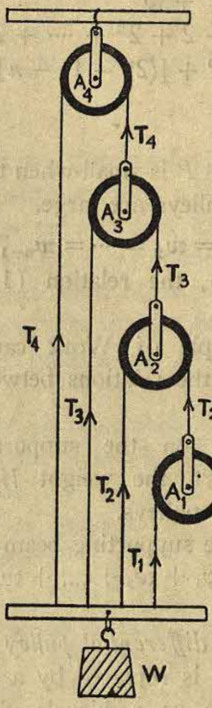


Fig. 24.23

For the equilibrium of the bar, we have

$$\begin{aligned} W &= T_1 + T_2 + T_3 + T_4 \\ &= (1 + 2 + 2^2 + 2^3)P + (1 + 2 + 2^2)w_1 \\ &\qquad + (1 + 2)w_2 + w_3 \\ &= (2^4 - 1)P + (2^3 - 1)w_1 \\ &\qquad + (2^2 - 1)w_2 + w_3. \end{aligned}$$

When there are n pulleys, the above result becomes

$$\begin{aligned} W &= (2^n - 1)P + (2^{n-1} - 1)w_1 \\ &\quad + (2^{n-2} - 1)w_2 + \dots + (2^2 - 1)w_{n-2} \\ &\quad + (2 - 1)w_{n-1} \qquad (3) \end{aligned}$$

If the weight of each of the movable pulleys be w, the relation (3) reduces to

$$\begin{aligned} W &= (2^n - 1)P + w\,[(2^{n-1} - 1) \\ &\quad + (2^{n-2} - 1) + \dots \\ &\quad + (2^2 - 1) + (2 - 1)] \\ &= (2^n - 1)P + w\,[(2 + 2^2 + \dots \\ &\quad + 2^{n-2} + 2^{n-1}) - (n - 1)] \end{aligned}$$

$$= (2^n - 1) P + w \, [(1 + 2 + 2^2 + \dots + 2^{n-1}) - n]$$

$$= (2^n - 1) P + [(2^n - 1) - n] \, w \qquad (4)$$

REMARKS

1. We find that P is small when the weights of the movable pulleys are large.

2. Putting $w_1 = w_2 = \dots = w_{n-1} = 0$ in (3) or $w = 0$ in (4), the relation (1) may be obtained.

3. The Principle of Work can also be applied to obtain the relations between P and W.

4. The stress on the supporting beam balances the effort, the weight W and the weights of all the pulleys.

$\therefore$ Stress on the supporting beam

$$= P + W + w_1 + w_2 + \dots + w_n.$$

24.6(*f*) *Weston's differential pulley*

Here the string is replaced by a long chain of the same type as a bicycle chain. The machine consists of two blocks; the upper block contains two concentric pulleys (of different radii R and r where $R > r$) while the lower block contains one pulley from which the weight W is suspended. An endless chain passes round the larger pulley of the upper block, then round the pulley of the lower block and next round the smaller pulley of the upper block.

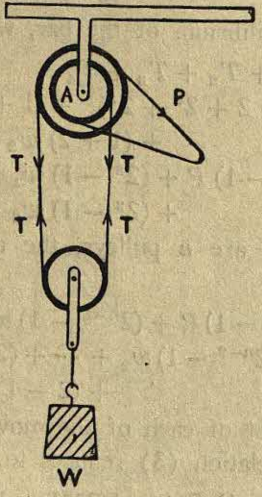


Fig. 24.24

The remainder of the portion of the chain is joined to the first portion of the chain.

The pulley grooves are replaced by cogs on to which the links of the chain fit. The upper pulley (the two taken together) is known as the differential pulley.

The effort P is applied as shown in Fig. 24.24. The length of the chain which passes over the larger pulley in the upper block is $2\pi R$ while that over the smaller pulley is $2\pi r$. Hence, when the differential pulley makes one complete revolution, the total length of the chain connecting the larger and smaller pulleys is diminished by $2\pi(R - r)$. But this decrease is divided equally between the two vertical lengths of the chain. Thus the weight W is raised through a vertical height of $\pi(R - r)$.

$\therefore$ Work done on $W = W \times \pi(R - r)$

and work done by $P = P \times 2\pi R$.

$\therefore$ By the Principle of Work, we have

$$P \times 2\pi R = W \times \pi(R - r)$$

or,

$$2PR = W(R - r) \qquad (1)$$

$$\therefore \text{ Mechanical Advantage} = \frac{W}{P} = \frac{2R}{R - r}.$$

REMARKS

1. We have assumed here that the portions of the chain hanging between the two blocks are vertical, and the weights of the chain and the pulleys are negligible.

2. The mechanical advantage may be made very great by making r nearly equal to R.

24.6(*g*) *Illustrative examples*

EXAMPLE 1

Find the least number of pulleys in a movable block weighing 10 Kg., in order that a weight of 120 Kg. may be lifted by a downward force not exceeding 28 Kg., and find the force.

Let n be the total number of portions of the string supporting the lower block (in the second system of pulleys) and P the required effort. The tension of each portion of the string is P. The n pulls in the strings have to support both the attached weight of 120 Kg. and the block weighting 10 Kg.

Therefore, $nP = (120 + 10)$ Kg. $= 130$ Kg.

Since $P \leqslant 28$ Kg., $nP \leqslant 28n$ Kg.

$\therefore \quad 130 \leqslant 28n$ so that $n \geqslant 4\frac{9}{14}$.

Hence $n = 5$ (taking the least value).

Hence five parts of the string must support the lower block. Therefore, the lower block must contain two pulleys and must have the end of the string attached to it. Also, putting $n = 5$, we have

$5P = 130$ Kg. ; $\therefore$ required force $P = 26$ Kg.

EXAMPLE 2

In the system of pulleys in which each hangs by a separate string, if there be 3 pulleys, it is found that a certain weight can be supported by an effort equal to 7 lbs. wt. ; but, if there be 4 pulleys, the same weight can be supported by an effort equal to 4 lbs. wt. Find the weight supported and the weight of the pulleys, which are equal.

Let W be the weight supported and w the weight of each pulley.

We have the relation (Sub-Section 24.6(d))

$$2^n P = W + (2^n - 1)w \qquad (1)$$

where P is the effort.

Given that $P = 7$ lbs. wt. when $n = 3$
and $P = 4$ lbs. wt. when $n = 4$.

Putting $n = 3$ and $P = 7$ in (1), we get

$$56 = W + 7w \qquad (2)$$

Putting $n = 4$ and $P = 4$ in (1), we get

$$64 = W + 15w. \qquad (3)$$

Subtracting (2) from (3), we have

$$8 = 8w \text{ so that } w = 1.$$

$\therefore$ From (2), $W = 56 - 7w = 56 - 7 = 49$.

$\therefore$ The weight supported $= 49$ lbs.

and the weight of each pulley $= 1$ lb.

EXAMPLE 3

In the third system of pulleys, a weight of 45 lbs. is supported by a force of 3 lbs. wt. The strings are attached to a 6 ft. bar from which the weight is suspended, and at distances of 1 ft. from each other. To what point of the bar should the weight be attached ? (The bar and the pulleys may be considered weightless).

Let there be n number of pulleys of which the upper one is fixed and $(n - 1)$ numbers are movable.

We are to use here the relation (*cf.* Sub-Section 24.6(e))

$$W = (2^n - 1)P.$$

Given that $W = 45$ lbs. wt., $P = 3$ lbs. wt.

$\therefore \quad 3(2^n - 1) = 45$; or, $2^n - 1 = 15$;

or, $2^n = 15 + 1 = 16 = 2^4$;

$\therefore \quad n = 4$.

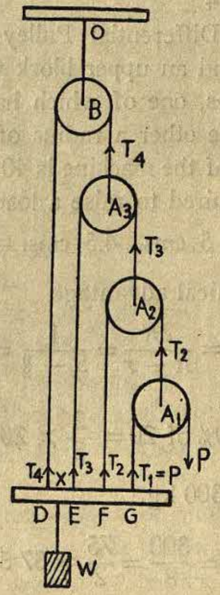


Fig. 24.25

Let the strings be attached to the bar at the points D, E, F, G as shown in Fig. 24.25, where $DE = EF = FG = 1$ ft. (given).

Let the point at which the weight is attached be X.

The resultant of T_1, T_2, T_3, T_4 must pass through X for the equilibrium of the system.

Also, we have

$$T_1 = P,\ T_2 = 2P,\ T_3 = 4P,\ T_4 = 8P.$$

Using the formula for determination of the centre of a system of parallel forces, we have

$$DX = \frac{T_4 \times O + T_3 \times DE + T_2 \times DF + T_1 \times DG}{T_4 + T_3 + T_2 + T_1}$$

$$= \frac{4P \times 1 + 2P \times 2 + P \times 3}{15P} = \frac{11}{15} < 1.$$

$\therefore$ $DX < DE$; i.e. X lies between D and E.

$\therefore$ The weight should be attached to the bar between the strings over the fixed pulley and the next one, and at a distance 11/15 ft. from the former.

EXAMPLE 4

A Weston Differential Pulley consists of a lower block and an upper block which has two cogged grooves, one of which has a radius of 5 cms. and the other a radius of 4.5 cms. and the efficiency of the machine is 40%. Calculate the effort required to raise a load of 300 kg.

Here $R = 5$ cms., 4.5 cms. $= 9/2$ cms.

The mechanical advantage

$$= \frac{2R}{R - r} = \frac{1}{5 - \frac{9}{2}} = 20.$$

$$\therefore\ \frac{W}{P} = 40\% \text{ of } 20 = \frac{2}{\cancel{5}} \times \overset{4}{\cancel{20}} = 8.$$

Now $W = 300$ kg.

$$\therefore\ P = \frac{W}{8} = \frac{300}{8} = \frac{75}{2} = 37{\cdot}5$$

Hence the required effort = 37.5 kgs. wt.

Exercise 24.4

1. In the first system of pulleys, if there be 4 weightless pulleys, what weight can be raised by an effort of 20 lbs. wt.?

2. Find the ratio of the power to the weight in that system of pulleys in which each movable pulley hangs by a separate string, the number of movable pulleys being 4.

If the weights of the pulleys be taken into account and are 1, 2, 3 and 4 lbs. respectively beginning with the highest, find what power will support a weight of 294 lbs.

3. If there be 4 movable pulleys whose weights, commencing with the lowest, are 4 kg., 5 kg., 6 kg. and 7 kg., what effort will support a weight of 112 kg. attached to the lowest pulley ?

4. If in a system of pulleys in which each string is attached to the weight, there will be four movable pulleys of weights 1, 2, 3 and 4 lbs. respectively, beginning with the lowest, find what power will support a weight of $4\frac{1}{4}$ cwt.

5. If there are 4 pulleys in the third system and each weighs 2 lbs., what weight can be raised by a power equal to the weight of 20 lbs.?

6. In the second system of pulleys, weights of 5 and 6 lbs. at the free end of the string support weights of 18 and 22 lbs. respectively at the lower block. Find the number of the strings and the weight of the lower block.

7. A man whose weight is 12 stone raises 3 cwt. by means of a system of pulleys in which the same rope passes round all the pulleys, there being 4 in each block and the rope being attached to the upper block ; neglecting the weights of the pulleys, find what will be his thrust on the ground if he pulls vertically downwards ?

8. A man weighing 15 stone, holding the bar to which the strings are attached in the third system of pulleys, can just raise a child weigh-

ing 1 stone, holding on to the last string. How many pulleys are there ?

9. A second system of pulleys has 5 pulleys in the upper block and 4 in the lower. How many times his own weight can a man raise by this machine, if each block weighs 1/10th of his own weight ?

10. A system consists of 4 pulleys, arranged so that each hangs by a separate string, one end being fastened to the upper block, and all the free ends being vertical. If the weights of the pulleys beginning from the lowest be w, $2w$, $3w$, and $4w$, find the power necessary to support a weight of $15w$, and the magnitude of the single force necessary to support the beam to which the other ends of the string are attached.

11. If there be three movable pulleys, each of mass 1 lb., in a system in which all the strings are parallel and are attached to the weight, and the force required to support a certain weight is half that which would be required if the pulleys were weightless, find that weight.

12. By the second system of pulleys having three pulleys in the lower block and the string attached to that block, a man standing on the ground supports a weight of 6 stone (including that of the lower block and the pulleys) and the pressure on the ground exerted by his feet is 128 lbs. wt. Find the maximum additional weight he can support.

13. If in a system of pulleys in which each hangs by a separate string there be three pulleys, each of mass 1 lb., and the power required to support a certain weight is twice that which would be required if these pulleys were weightless, find the weight.

14. It is required to lift a weight of 10 cwt. with 4 pulleys each weighing 8 lbs. Would you prefer the first or the second system as being more advantageous ?

15. If the strings passing round a system of 4 weightless pulleys be fastened to a rod without weight at distances successively an inch apart, find to what point of the rod the weight must be attached so that the rod may be horizontal.

16. In the third system in which there are 4 pulleys of equal size (of which one is fixed), each of weight 1 lb., find the effort required to support a weight of 161 lbs. Also find to what point of the bar the weight must be attached, so that the beam may remain horizontal.

17. Being given four weightless pulleys, find the greatest weights that can be supported by a force of 1 lb. wt. when the pulleys are arranged as in the first, second and third systems respectively in such a way that the effort is applied downwards.

18. In the first system of four heavy pulleys, if P be the effort and W the weight, show that the stress on the beam is intermediate between $15/16\ W$ and $15\ P$.

19. In the second system of pulleys, a platform is suspended from the lower block. A man of weight W, standing on the platform, supports himself by exerting on the string a force equal to P. If n be the total number of pulleys in the system, and mW the weight of the platform and the lower block together, show that $\dfrac{W}{P} = \dfrac{n+1}{m+1}$.

20. In the system of pulleys in which each string is attached to the weight, if each pulley have a weight w, and the sum of the weights of the pulleys be W', and P, W be respectively the effort and the weight in this case, show that the effort $P + w$ would support the weight $W + W'$ in the system if the pulleys had no weight.

21. In the first system of pulleys, in which there are three movable pulleys, the weights of the pulleys beginning from the highest one increase in arithmetical progression downwards,

and a power P supports a weight W. The pulleys are then arranged in the reverse order, the highest being placed lowest and it is found that the interchange of P and W maintains equilibrium. Prove that $3(W+P)=2W_1$ where $W_1=$ total weight of the three pulleys.

22. If in the second system of pulleys the working tension of a rope is reduced by the factor f in passing over a pulley (in consequence of resistances due to friction etc.); prove that the mechanical advantage is $(f-f^{n+1})/(1-f)$.

23. If there be n weightless pulleys and if a string whose ends are attached to the weights P and W, carry a pulley from which a weight W' is suspended, find the relation among P, W and W'.

24. If the third system of pulleys is modified by making the string that passes over each pulley A pass round a small pulley attached to the weight, and the string is then fastened to the pulley A, show that if the weights of the pulleys be neglected, and the number of separate strings be n, the mechanical advantage will be (3^n-1).

25. In the first system of pulleys, if the weights of the n pulleys, reckoning from the one nearest to W, increase in a geometric progression, the common ratio of which is 2, prove that $P=\frac{W}{2^n}+\frac{w}{3}(2^n-2^{-n})$ where w is the weight of the lowest pulley.

26. In the second system of pulleys, if a basket be suspended from the lower block and a man in the basket supports himself and the basket, by pulling at the free end of the rope, find the tension he exerts, neglecting the inclination of the rope to the vertical, and assuming the weight of the man and the basket to be W.

If the free end of the rope passes round a pulley attached to the ground and then be held by the man, find the force he exerts.

27. In the third system if there be n pulleys, each of diameter $2a$ and of negligible weight, show that the distance of the point of application of the weight from the line of action of the effort is $\frac{2^n}{2^n-1}na$.

28. A weight W is supported by a small frictionless pulley which runs on a string of length l, whose extremities are attached to two fixed points P and Q. Given that $PQ=d$, and that the inclination of PQ to the horizon is α. Find the tension of the string.

29. There are three movable pulleys of weights w_1, w_2 and w_3 in the third system, and the force P then balances a load W; when the first and second pulleys are interchanged, then a force P' balances the same load. Show that

$$\frac{P-P'}{w_2-w_1}=\frac{4}{15}.$$

30. In a system of n weightless pulleys, each hanging by separate strings whose fixed ends are parallel and make an angle θ with the vertical, show that the mechanical advantage is $\sin 2^n\theta/\sin\theta$.

31. If in the first system of pulleys, P is the power (acting upwards), W the weight, and R the stress on the beam from which the pulleys hang, show that $(1-2^{-n})W<R<(2^n-1)P$, n being the number of pulleys in the system.

32. In the third system of n pulleys, if the weight of the lowest pulley be equal to the power P, of the next lowest to $3P$, and so on, that of the highest movable pulley being $3^{n-2}P$, prove that $\frac{W}{P}=\frac{3^n-1}{2}$.

33. A weight is supported as follows: n movable pulleys and a fixed one arranged as in the third system, and the string which passes over the lowest pulley of the system passes round a pulley fixed to the weight, and is then attached to the lowest of a set of n movable pulleys arranged as in the first system. All the

strings being vertical and all the pulleys of weight w, find the power required to raise a weight W.

34. In the third system of n movable pulleys, all being weightless, if the free end of the string round the lowest pulley be attached to the bar from which the weight is suspended, show that the tension of the string is diminished in the ratio $(2^n - 1) : 2^n$.

35. Prove that in a system of three light pulleys, of which the first is fixed and each of the others is supported by the string of the preceding one, the strings being all parallel and all attached to the weight W, the downward acceleration of this weight, when the power P is not sufficient to balance it, is

$$\frac{W - 7P}{W} g,$$

where g is the acceleration due to gravity.

ANSWERS TO EXERCISES

Differential Calculus

Exercise 1.1

1. $V=\frac{4}{3}\pi r^3$; V is an explicit function of r.
2. $t=s/0{\cdot}6$ for $0\leqslant s\leqslant 45$
   $t=75+s/0{\cdot}9$ for $45\leqslant s\leqslant 90$.
3. $1, \frac{1}{2}, 1, \frac{2}{3}$. 4. $\sqrt{2}, 2$. 6. $\frac{1}{2}\log 2$.
7. (*i*) $-1\leqslant x\leqslant 1$ (*ii*) $x>0$
   (*iii*) $-\infty<x<+\infty$
   (*iv*) $k\pi\leqslant x\leqslant k\pi+\pi/3\ (k=0, \pm 1, \pm 2, \cdots)$
   (*v*) $x\neq 2$. (*vi*) $-2\leqslant x\leqslant 5$.
8. $\phi(x)=2x^4-5x^2-10$, $\psi(x)=-3x^3+6x$
9. (*i*) Even (*ii*) odd (*iii*) odd.
10. *Hint*: Use $f(x)=\frac{1}{2}[f(x)+f(-x)]$
    $+\frac{1}{2}[f(x)-f(-x)]$
11. $x=-\dfrac{\log(1-y)}{\log 2}$; $x=5\pm\sqrt{9-y}$
14. (*i*) Periodic, $T=2\pi$ (*ii*) Periodic, $T=\pi$
    (*iii*) Periodic, $T=\dfrac{2\pi}{n}$ (*iv*) Periodic, $T=\dfrac{2\pi}{3}$
    (*v*) Non-periodic
16. $x=y$ when $-\infty<y\leqslant 0$;
    $x=\sqrt{y}$ when $0<y<+\infty$.

Exercise 2.1

1. (*i*) 0 (*ii*) $3x^2$ (*iii*) -1 (*iv*) $\frac{4}{3}$ (*v*) $\frac{1}{9}$
(*vi*) $-\frac{1}{58}$ (*vii*) $\dfrac{1}{2\sqrt{x}}$ (*viii*) $-\frac{1}{3}$ (*ix*) $\frac{1}{2}\sin 2$
(*x*) 0 (*xi*) $\dfrac{2}{\pi}$ (*xii*) $\frac{5}{2}$ (*xiii*) 0 (*xiv*) $\frac{1}{2}$
(*xv*) $\cos a$ (*xvi*) $\frac{1}{2}(n^2-m^2)$ (*xvii*) $\frac{1}{4}$ (*xviii*) 1
(*xix*) 1 (*xx*) $10\log e$ (*xxi*) $a-b$ (*xxii*) $-\frac{1}{2}$
(*xxiii*) 1 (*xxiv*) $\log a$ (*xxv*) 1
(*xxvi*) $\frac{1}{12}$.
2. 2 and 3.

Exercise 2.2

2. (*i*) discontinuous (*ii*) continuous
   (*iii*) continuous.
4. (*i*) $x=2, 4$ (*ii*) none (*iii*) none (*iv*) 6, 2.
7. (*i*) $f(0)=n$ (*ii*) $f(0)=\frac{1}{2}$ (*iii*) $f(0)=2$
(*iv*) $f(0)=2$ (*v*) $f(0)=0$ (*vi*) $f(0)=1$

Exercise 3.1

1. (*i*) 3 (*ii*) 0·21 (*iii*) $2h+h^2$.
2. (*i*) 0·1 (*ii*) -3 (*iii*) $\sqrt[3]{a+h}-\sqrt[3]{a}$.
3. (*i*) 624 (*ii*) 1560.
4. (*i*) -1 (*ii*) 0·1 (*iii*) $-h$, 0.
5. 21. 6. 15 cm/sec. 7. 7·5.
8. $f'(x)=\underset{\Delta x\to 0}{Lt}\ \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$.
9. (*i*) $\dfrac{\Delta T}{\Delta t}$ (*ii*) $\dfrac{dT}{dt}=\underset{\Delta t\to 0}{Lt}\ \dfrac{\Delta T}{\Delta t}$ where T is temperature at time t. 10. $\sec^2 x$. 11. (*i*) $3x^2$
(*ii*) $-\dfrac{2}{x^3}$ (*iii*) $\dfrac{1}{2\sqrt{x}}$ (*iv*) $-\dfrac{1}{\sin^2 x}$. 12. $\dfrac{1}{12}$.
13. $f'(0)=-8$, $f'(1)=0$, $f'(2)=0$.
14. $x_1=0$, $x_2=3$. 15. 30 m/sec. 16. 1, 2.
17. -1. 18. $-1, 2, \tan\theta=3$.
19. $y=x$. 20. $\frac{1}{8}$.
21. (*i*) $ma\,x^{m-1}+b(m+n)x^{m+n-1}$
(*ii*) $\dfrac{4b}{3x^2\sqrt[3]{x}}-\dfrac{2a}{3x\sqrt[3]{x^2}}$ (*iii*) $\dfrac{bc-ad}{(c+dx)^2}$
(*iv*) $\dfrac{1-4x}{x^2(2x-1)^2}$ (*v*) $\dfrac{1}{\sqrt{x}(1-\sqrt{x})^2}$ (*vi*) $\dfrac{4}{\sin^2 2x}$
(*vii*) $\dfrac{-2}{(\sin x-\cos x)^2}$ (*viii*) $x^2\sin x$ (*ix*) $x\,e^x$
(*x*) $\dfrac{x(2\log x-1)}{(\log x)^2}$ (*xi*) $\dfrac{5x^4-x^5}{e^x}$ (*xii*) $3x^2\log x$
(*xiii*) $\dfrac{2}{x}+\dfrac{\log x}{x^2}-\dfrac{2}{x^2}$ (*xiv*) $e^x(\cos x-\sin x)$.
23. (*i*) yes (*ii*) yes, 0. 26. (*i*) 0 (*ii*) 0
(*iii*) yes, 0. 28. $(1, -3)$. 29. $m=-\frac{1}{11}$.
30. $x-4y+4=0$, $4x+y-18=0$.

Exercise 3.2

1. (*i*) $\dfrac{3a}{c}\left(\dfrac{ax+b}{c}\right)^2$ (*ii*) $16x(3+2x^2)^3$
(*iii*) $\dfrac{-x}{\sqrt{1-x^2}}$ (*iv*) $-\sqrt{\sqrt[3]{\dfrac{a^2}{x^2}}-1}$
(*v*) $-10\cos x(3-2\sin x)^4$ (*vi*) $\dfrac{1-\tan^2 x+\tan^4 x}{\cos^2 x}$
(*vii*) $\dfrac{-1}{2\sin^2 x\sqrt{\cot x}}$ (*viii*) $\dfrac{-16\cos 2x}{\sin^3 2x}$

(*ix*) $\frac{2\cos x}{3\sqrt[3]{\sin x}}+\frac{3\sin x}{\cos^4 x}$ (*x*) $\frac{e^x+xe^x+1}{2\sqrt{xe^{x+x}}}$

(*xi*) $\frac{2e^x-2^x\log 2}{3\sqrt[3]{(2e^x-2^x+1)^2}}+\frac{5\log^4 x}{x}$

(*xii*) $3\cos 3x-\frac{1}{5}\sin\frac{x}{5}+\frac{1}{2\sqrt{x}\cos^2\sqrt{x}}$

(*xiii*) $(2x-5)\cos(x^2-5x+1)-\frac{a}{x^2\cos^2\frac{a}{x}}$

(*xiv*) $\sin(2t+\phi)$ (*xv*) $-2\frac{\cos x}{\sin^3 x}$ (*xvi*) $\frac{-2}{x\sqrt{x^4-1}}$

(*xvii*) $\frac{-1}{2\sqrt{x-x^2}}$ (*xviii*) $2x\,10^{2x}(1+x\log 10)$

(*xix*) $\cot x\log e$ (*xx*) $\frac{-2x}{1-x^2}$

(*xxi*) $\sin 2^t+2^t t\cos 2^t\log 2$

(*xxii*) $\frac{1}{(1+\log^2 x)x}+\frac{1}{(1+x^2)\tan^{-1}x}$

(*xxiii*) $\frac{2\log x}{x}-\frac{1}{x\log x}$

(*xxiv*) $\frac{1}{2x\sqrt{\log x+1}}+\frac{1}{2(\sqrt{x}+x)}$ (*xxv*) $\frac{a}{2}e^{\sqrt{ax}}$

(*xxvi*) $\sin 2x\,e^{\sin^2 x}$ (*xxvii*) $e^{\alpha t}(\alpha\cos\beta t-\beta\sin\beta t)$

(*xxviii*) $\frac{2x+1}{x^2-x-2}$.

2. (*i*) $y_1=1 \quad x>0$
$\quad =-1 \quad x<0$
$y_1(0)$ does not exist. (*ii*) $y_1=|2x|$

3. $\frac{1}{x}$. 4. $f'(x)=\begin{cases}-1, & x\leqslant 0\\ -e^{-x}, & x>0.\end{cases}$ 5. -1.

6. $\frac{1}{2}+\frac{\sqrt{3}}{3}$. 7. 6π. 8. $1-x$. 9. -1.

12. (*i*) $(1+2x)(1+3x)+2(1+x)(1+3x)+3(x+1)(1+2x)$

(*ii*) $\sqrt[x]{x}\,\frac{1-\log x}{x^2}$ (*iii*) $-\frac{(x+2)(5x^2+19x+20)}{(x+1)^4(x+3)^5}$

(*iv*) $x^{\sqrt{x}-\frac{1}{2}}(1+\frac{1}{2}\log x)$ (*v*) $\frac{x^2-4x+2}{2\sqrt{x(x-1)(x-2)^3}}$

(*vi*) $x^{x^x}x^x\left(\frac{1}{x}+\log x+\log^2 x\right)$

(*vii*) $\frac{3x^2+5}{3(x^2+1)}\sqrt[3]{\frac{x^2}{x^2+1}}$

(*viii*) $x^{\sin x}\left(\frac{\sin x}{x}+\cos x\log x\right)$

(*ix*) $(\cos x)^{\sin x}(\cos x\log\cos x-\sin x\tan x)$

(*x*) $\left(1+\frac{1}{x}\right)^x\left[\log\left(1+\frac{1}{x}\right)+\frac{1}{1+x}\right]$

(*xi*) $x^x(1+\log x)$ (*xii*) $x^{x^2+1}(1+2\log x)$

(*xiii*) $\left(\tan^{-1}x\right)^x\left[\log\tan^{-1}x+\frac{x}{(1+x^2)\tan^{-1}x}\right]$

13. (*i*) $\frac{3}{2}t^2$ (*ii*) $-\frac{b}{a}$ (*iii*) $\frac{t-1}{t+1}$

(*iv*) $-\frac{b}{a}\tan t$ (*v*) $\frac{-2t}{1-t^2}$ (*vi*) $-\tan 3t$

(*vii*) $\frac{t(2-t^3)}{1-2t^3}$ (*viii*) $y_1=\begin{cases}-1, & t<0\\ 1, & t>0\end{cases}$ (*ix*) $\frac{2}{3\sqrt[6]{t}}$

(*x*) $-2e^{3t}$ (*xi*) $\frac{t+1}{t(t^2+1)}$ (*xii*) $\tan t$.

14. 1. 15. 1. 16. ∞. 18. No.

19. Yes, since the equality is an identity.

20. (*i*) -1 (*ii*) $-\frac{b^2x}{a^2y}$ (*iii*) $\frac{y\cos^2 y}{1-x\cos^2 y}$

(*iv*) $-\frac{x^2}{y^2}$ (*v*) $\frac{y}{x}\,\frac{1-x^2-y^2}{1+x^2+y^2}$ (*vi*) $-\frac{x(3x+2y)}{x^2+2y}$

(*vii*) $(x+y)^2$ (*viii*) $\frac{1}{x+y-1}$ (*ix*) $\frac{y}{x}+e^{\frac{y}{x}}$

(*x*) $\frac{y}{x-y}$ (*xi*) $\frac{1-y^3}{1+3xy^2+4y^3}$

(*xii*) $\frac{x\log y-y}{y\log x-x}\cdot\frac{y}{x}$.

21. -1, 22. $\frac{1}{2}$. 23. 0.

24. (*i*) $56x^6+210x^4$ (*ii*) $\frac{-x}{\sqrt{(a^2+x^2)^3}}$

(*iii*) $e^{x^2}(4x^2+2)$ (*iv*) $2\cos 2x$

(*v*) $2\tan^{-1}x+\frac{2x}{1+x^2}$ (*vi*) $\frac{2}{1-x^2}+\frac{2x\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}$

(*vii*) $\frac{2(1-x^2)}{3(1+x^2)^2}$. 27. $\frac{2}{(1+x)^3}$.

28. $16\sin(2\pi+2x)$. 30. (*i*) $\underline{|n}\,;\ (1-x)^{-(n+1)}$

(*ii*) $(-1)^{n+1}\frac{1.\,3.\,\cdots(2n-3)}{2^n x^{n-\frac{1}{2}}}$

31. (*i*) $\sin\left(x+\frac{n\pi}{2}\right)$ (*ii*) $\frac{(-1)^{n+1}\underline{|n}}{(1+x)^{n+1}}$

(*iii*) $2^n\cos\left(2x+\frac{n\pi}{2}\right)$ (*iv*) $\frac{2\underline{|n}}{(1-x)^{n+1}}$ (*v*) $(-3)^n e^{-3x}$

(*vi*) $2^{n-1}\sin\left[2x+(n-1)\frac{\pi}{2}\right]$

(*vii*) $(-1)^{n-1}\frac{\underline{|n-1}}{(1+x)^n}$ (*viii*) $\frac{(-1)^{n-1}\underline{|n-1}\,a^n}{(ax+b)^n}$

32. (*i*) xe^x+ne^x

(*ii*) $2^{n-1}e^{-2x}$

$$\left[2(-1)^n x^2+2n(-1)^{n-1}x+\frac{n(n-1)}{2}(-1)^{n-2}\right]$$

(*iii*) $\frac{(-1)^n 6\underline{|n-4}}{x^{n-3}}$ for $n\geqslant 4$

(*iv*) $\frac{(-1)^{n-1}1-3\cdots(2n-3)}{2^n x^{\frac{2n+1}{2}}}\left[x-(2n-1)\right]$

(*v*) $(1-x^2)\cos\left(x+\frac{n\pi}{2}\right)-2nx\cos\left(x+\frac{(n-1)\pi}{2}\right)$
$-n(n-1)\cos\left(x+\frac{(n-2)\pi}{2}\right)$.

33. $y_n(0)=\underline{|n-1}$. 34. (*i*) $9t^3$ (*ii*) $2t^2+2$

(*iii*) $\dfrac{-1}{a\sin^3 t}$ (*iv*) 0 (*v*) $2e^{3at}$ (*vi*) $t\dfrac{1+t}{(1-t)^3}$.

35. $\dfrac{-2e^{-t}}{(\cos t+\sin t)^3}$. 36. (*i*) $\dfrac{2\cot^4 t}{\sin t}$

(*ii*) $-6e^{3t}(1+3t+t^2)$ (*iii*) $\dfrac{4e^{2t}(2\sin t-\cos t)}{(\sin t+\cos t)^5}$.

37. $-\dfrac{1}{y^3}$. 38. (*i*) $-\dfrac{p^2}{y^3}$ (*ii*) $-\dfrac{b^4}{a^2y^3}$.

39. $\dfrac{d^2y}{dx^2}=\dfrac{y}{(1-y)^3}$, $\dfrac{d^2x}{dy^2}=\dfrac{1}{y^2}$.

40. Velocity = 5 ; 4·997 ; 4·7.
Acceleration = 0 ; − 0·006 ; − 0·6.

Exercise 3.3

1. $\Delta y=0.009001$, $dy=0.009$. 2. 1.

3. $dy=\dfrac{1}{2700}\doteq 0.00037$. 4. $dy=\dfrac{\pi}{45}\doteq 0.0698$.

5. 0.515. 6. (*i*) 0.485 (*ii*) 0.965

(*iii*) − 0.045. (*iv*) 1.1 (*v*) 0.81

Integral Calculus

Exercise 4.1

1. (*i*) $\dfrac{10}{3}x^{\frac{3}{2}}-60\sqrt{x}-\dfrac{90}{\sqrt{x}}+c$ (*ii*) $\tan x-x+c$

(*iii*) $\dfrac{e^{4x}}{4}+c$ (*iv*) $2\left(\sin\dfrac{x}{2}-\cos\dfrac{x}{2}\right)+c$

(*v*) $\sin x-\frac{1}{3}\sin^3 x+c$

(*vi*) $-\frac{1}{8}\cos 2x-\frac{1}{16}\cos 4x+\frac{1}{24}\cos 6x+\dfrac{c}{2}$

(*vii*) $\tan x-\sec x+c$ (*viii*) $\dfrac{x^3}{3}-\dfrac{3}{2}x^2+3x+c$

(*ix*) $-\dfrac{1}{\log\alpha}\left[\dfrac{1}{3\alpha^{3x}}+\dfrac{1}{2\alpha^{2x}}+\dfrac{1}{\alpha^x}\right]+c$

(*x*) $\tan x-\cot x+c$.

Exercise 5.1

1. $\frac{1}{2}\tan^{-1}(x^2)+c$.

2. $\dfrac{1}{(n+2)}(x^2+2x+3)^{n+1}+c$.

3. $\frac{1}{2}\sin^{-1}x-\sqrt{1-x^2}\left(\ +\dfrac{x}{2}\right)+c$.

4. $\frac{1}{2}\log(3+2x+x^2)+c$.

5. $\frac{1}{2}\tan^{-1}\{\frac{1}{2}(1+e^x)\}+c$. 6. $\sin^{-1}\left(\dfrac{x-a}{a}\right)+c$

7. $x-\log(x^2+x+1)+\dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)+c$.

8. $\dfrac{1}{2}\log\left\{\dfrac{\sqrt{4x+3}-1}{\sqrt{4x+3}+1}\right\}+c$. 9. xe^x-e^x+c.

10. $x\tan^{-1}x-\frac{1}{2}\log(1+x^2)+c$.

11. $\dfrac{e^{2x}}{2}\left\{\dfrac{1}{\sqrt{20}}\sin(4x-\tan^{-1}2)\right.$

$\left.+\dfrac{1}{\sqrt{8}}\sin\left(2x-\dfrac{\pi}{4}\right)\right\}+c$.

12. $x(\log x)^{-1}+c$.

13. $(x+a)\tan^{-1}\left(\dfrac{x}{a}\right)^{\frac{1}{2}}-\sqrt{ax}+c$.

14. $\dfrac{1}{2}e^x\left\{1-\dfrac{1}{5}(\cos 2x+2\sin 2x)\right\}+c$.

15. $\sin x\cos x\log(1+\tan x)-\frac{1}{2}x$
$+\frac{1}{2}\log(\sin x+\cos x)+c$.

Exercise 6.1

1. (*i*) e^b-e^a (*ii*) $\dfrac{1}{k}(e^{-ka}-e^{-kb})$ (*iii*) $\dfrac{1}{3}$

(*iv*) $\dfrac{a}{4}+b$. 2. (*i*) $\dfrac{\pi}{4}$ (*ii*) $\dfrac{\pi a^2}{2}$ (*iii*) $\dfrac{2}{3}$ (*iv*) $\dfrac{1}{4}$.

3. (*i*) $1+\dfrac{\pi}{2}$ (*ii*) $\log\left\{\dfrac{1+e^{-1}}{1+e^{-2}}\right\}$ (*iii*) $\dfrac{\pi}{2}$

(*iv*) $2(\sqrt{3}-1)+2\log\left(1+\dfrac{2}{\sqrt{3}}\right)$ (*v*) 0.

Exercise 7.1

1. (*i*) $\log\left(\dfrac{b^b}{a^a}\right)+a-b$

(*ii*) $\dfrac{\pi ab}{4}-\dfrac{b^2}{2a}\sqrt{a^2-b^2}-\dfrac{ab}{2}\sin^{-1}\left(\dfrac{\sqrt{a^2-b^2}}{a}\right)$

(*iii*) $\frac{4}{3}a^{\frac{3}{2}}$.

2. 2. 3. 72. 4. πab.

5. 3·2. 6. $a^2\left(\dfrac{\pi}{4}-\dfrac{2}{3}\right)$.

Differential Equations

Exercise 8.1

1. (*i*) $x+y+\log xy=c$

(*ii*) $y=c+a\tan^{-1}\left[\dfrac{x+y}{a}\right]$

(*iii*) $e^x+e^{-y}+\dfrac{x^3}{3}=c$ (*iv*) $\tan x\tan y=c$

(*v*) $\tan(x+y)-\sec(x+y)=x+c$

(*vi*) $3\log(x^2+y^2)=4\tan^{-1}\left(\dfrac{y}{x}\right)+c$.

(*vii*) $\frac{2}{3}\log\frac{y}{x}+\frac{1}{6}\log\left(\frac{y^2-x^2}{y^2}\right)=c-\log x$

(*viii*) $x\left[\frac{y^3}{x^3}+\frac{3y}{x}+2\right]^{\frac{1}{3}}=c$

(*ix*) $y+\sqrt{x^2+y^2}=cx^2$ (*x*) $\sin\left(\frac{y}{x}\right)=cx$

(*xi*) $(\alpha+\beta)\log\{y-x+1\}$
$+(\alpha-\beta)\log\{y+x-1\}=c$

(*xii*) $\sqrt{3}\log[Y^2-2XY-2X^2]$
$+2\log\left[\frac{[Y-(1+\sqrt{3})X]}{[Y-(1-\sqrt{3})X]}\right]=c$
where $X=x-1$ and $Y=y-2$

(*xiii*) $\log(x-2)+c$
$=-\frac{1}{2}\log\left\{2\left(\frac{y-1}{x-2}\right)-\left(\frac{y-1}{x-2}\right)^2+1\right\}$

(*xiv*) $x=\frac{1}{2}(x+y)-\frac{1}{4}\log(2x+2y-1)+c$

(*xv*) $3x^2+4xy+3y^2=10x+10y-c.$

Exercise 9.1

1. (*i*) $y=c_1e^{-x}+c_2e^{-2x}$ (*ii*) $y=c_1e^{-ax}+c_2e^{-bx}$
(*iii*) $y=(A+Bx)e^{2x}$ (*iv*) $y=c_1+c_2e^{-2x}$
(*v*) $y=c_1e^{x}+c_2e^{2x}$
(*vi*) $y=A\cos\sqrt{c}\,x+B\sin\sqrt{c}\,x$
(*vii*) $y=e^{-\frac{x}{2}}\left[A\cos\frac{\sqrt{3}}{2}x+B\sin\frac{\sqrt{3}}{2}x\right]$
(*viii*) $y=e^{\frac{x}{2}}\left\{A\cos\frac{x}{2}+B\sin\frac{x}{2}\right\}$
(*ix*) $y=e^{x}\{A\cos\sqrt{3}x+B\sin\sqrt{3}x\}$
(*x*) $y=Ae^{\left(\frac{3+\sqrt{5}}{2}\right)x}+Be^{\left(\frac{3-\sqrt{5}}{2}\right)x}$

2. (*i*) $y=-\frac{1}{8}e^{-6x}+\frac{1}{8}e^{-2x}$ (*ii*) $y=1-e^{-2x}$
(*iii*) $x=a\cos\mu t$ (*iv*) $u=(3+8x)e^{-x}$

Exercise 9.2

1. (*i*) $(1+y)(1+e^{x})=c\,e^{y}$
(*ii*) $y=(A+Bx)e^{-\frac{2}{3}x}$ (*iii*) $(x^2+1)(y^2+1)=c$
(*iv*) $y=c_1e^{-\frac{4}{3}x}+c_2e^{-\frac{3}{2}x}$ (*v*) $\log x=\cos\left(\frac{y}{x}\right)+c$
(*vi*) $\log(2x+y-1)+x+2y=c.$
2. (*i*) $y=2x\,e^{3(1-x)}$ (*ii*) $y=e^{-3x}\sin 4x.$
3. $u=e^{At}\{A\cos\mu t+B\sin\mu t\}.$
4. $\sqrt{v-t}+\log\left\{\sqrt{v-t}-1\right\}=\frac{t}{2}+c$
5. $x=c_1e^{\left\{\frac{-k+\sqrt{k^2-4\mu}}{2}\right\}t}+c_2e^{\left\{\frac{-k-\sqrt{k^2-4\mu}}{2}\right\}t}$

Dynamics

Exercise 11.1

1. 14 km.p.h., 10 km.p.h. 2. 17 cm. 3. 20 cm./sec., 4 cm./sec. 4. 60°. 5. 120°. 6. 120°. 7. 100 ft./sec. 8. At angle $\tan^{-1}5/44$ with the direction of the train. 9. 120 ft./min. 10. 120°. 11. 120°, 90° with the current. 12. At angle of $\cos^{-1}(-3/5)$ with the current ; perpendicular to the current so that his resultant direction makes an angle of $\tan^{-1}(5/3)$ with the current. 13. 2 hr. 45 min. 14. 50 sec. ; 200 ft.

Exercise 11.2

1. $(\sqrt{3}-1)u$; $(\sqrt{6}-\sqrt{2})\frac{u}{2}$. 2. $5\sqrt{3}$ and 5 cm./sec.
3. $7\sqrt{3}$ and 14 ft./sec. 4. $\sqrt{50+32\sqrt{2}}$ ft./sec. at an angle of $\tan^{-1}\frac{7+4\sqrt{2}}{17}$ with the first velocity.
5. 5 ft./sec. at an angle $\tan^{-1}\frac{3}{4}$ with the horizontal.
6. $7u$ at $\cos^{-1}\frac{11}{14}$ with the third velocity.
7. 164 ft. nearly, in a direction approximately at an angle $\tan^{-1}\frac{3}{8}$ S. of E.

Exercise 11.3

2. 4 m.p.h. from N., wind appears to blow from the N.E. at $4\sqrt{2}$ m.p.h.
3. 55 ft./sec. at an angle $\tan^{-1}(-\frac{3}{4})$ with the direction of the train's motion.
4. 12 m.p.h. towards the west.
5. $\cot^{-1}(3\sqrt{2}-1)$ east of south.
6. 45° with the vertical.
7. The cyclist is moving S.W. at $8\sqrt{2}$ m.p.h.
8. The true velocity of the wind is $10\sqrt{2-\sqrt{2}}$ m.p.h. at an angle $22\frac{1}{2}$° S. of E.
9. 30 m.p.h.
10. The true velocity of the wind is $15\sqrt{5}$ m.p.h. at an angle $\tan^{-1}\frac{1}{2}$ E. of S.
11. $2\sqrt{2}$ m.p.h. at an angle of 45° with the vertical.
12. 6 m.p.h. at an angle of 30° with the vertical.
13. $4\sqrt{2}$ m.p.h. from N.W.
14. 39 m.p.h. in a direction $\cos^{-1}(\frac{5}{12})$ N.E.
15. $40\sqrt{5}$ yards ; 33 secs.
16. 50 yards each.
17. $\frac{3}{4}$ hour, $\frac{45}{4}\sqrt{2}$ miles.

Exercise 12.1

1. 5 sec., 62.5 cm. 2. 14 ft./sec^2. 3. 41 ft.
4. $9\frac{2}{3}$ ft./sec., $2\frac{2}{3}$ ft./sec^2. 5. 30 ft./sec., 16 ft./sec^2.
6. 10 yds. 7. 1 in. 8. 7 ft./sec., 4 ft./sec^2.
9. 30 ft./sec., -2 ft./sec.^{2}. 10. $16\frac{7}{8}$ miles.
11. 8 secs., 16 ft./sec.^{2}. 12. $16\frac{7}{8}$ miles.
13. 8 secs., 16 ft./sec. 18. 8 secs.
19. $\frac{14u}{3}$ cm. 21. $v=50e^{-\frac{1}{2}t}$, $x=100\left(1-e^{-\frac{1}{2}t}\right)$

23. $v_0 = 0, f_0 = 0,$; $v_5 = \frac{15}{8}$ m./min., $f_5 = 45$ m./min^2.

24. 3.86 m./sec^2. 25. $\frac{4b - c - 3a}{2n}$.

Exercise 12.2

1. (*i*) 1 sec., $2\frac{3}{4}$ sec. (*ii*) 4 sec.
2. (*i*) $\frac{15}{32}$ sec. (*ii*) $1\frac{3}{4}$ sec.
3. 5 sec. 4. (*i*) 3 sec. (*ii*) 16 ft.
5. 1024 ft. 6. 400 ft. 7. 264 ft.
8. 2 sec. or $5\frac{1}{2}$ sec. 9. 3 sec.
10. 1 sec., 16 ft./sec., $1\frac{1}{4}$ sec.
12. 1 ft. above the top of the window.
13. $32\frac{7}{8}$ ft., 29.2 ft./sec. 14. 10 sec.
15. 1200 ft./sec. 16. 784 ft. 17. $1\frac{1}{4}$ sec.
18. 100 ft., 4 sec. after first body started ; 48, 0 ft./sec.
19. Between 2 and 3 sec. after projection ; 112 ft./sec.
20. $1\frac{1}{8}$ sec. 22. $(n - \frac{1}{2})g$, $\frac{1}{2}(n - \frac{1}{2})^2 g$.
25. $2\sqrt{H/g}$, $27\, gt^2/16$.

Exercise 13.1

1. (*i*) 200 poundals (*ii*) 6.25 lbs. wt. 2. 1 lb. wt.
3. 480 ft. 5. 180 ft.
6. (*i*) $7\frac{1}{16}$ lbs. wt. (*ii*) $237\frac{3}{8}$ lbs. wt.
7. 1020 lbs. wt. 8. 242 tons wt.
9. $363\frac{1}{3}$ cm./sec. ; $181\frac{2}{3}$ cm. ; 21800 cm.
10. 440 ft. ; 27.5 tons wt.
12. 31250 poundals ; .002 sec. 13. $133\frac{1}{3}$ ft./sec.
14. $1\frac{1}{14}$ in. 15. 605 ft. 16. $10778\frac{1}{3}$ lbs. wt.
17. 19 cm./sec^2. 18. 0.
19. (*i*) $192\frac{1}{2}$ lbs. wt. (*ii*) $87\frac{1}{2}$ lbs. wt. 20. 24 lbs.
21. (*i*) 10 lbs. (*ii*) $11\frac{9}{16}$ lbs. wt..
22. 9 lbs. ; $g/9$. 23. $g/13$ downwards.

Exercise 14.1

1. $\ddot{x} = -k^2 x$, S.H.M.
2. $\frac{a\sqrt{3}}{2}$ (*a* being the amplitude).
7. nearly 1600.
11. (*i*) $\frac{4\pi^2}{T^2}(a^2 - x^2)$ (*ii*) $\frac{4\pi^2 a^2}{T^2} \sin^2 \frac{2\pi t}{T^2}$.

Exercise 15.1

1. $c^2 x = ay^2 + bc^2$ which is a parabola.
4. $\theta^2 r$; $\tan^{-1}\left(\frac{b}{a} \tan \theta t\right)$ with the x-axis.
5. θ. 6. $f = 4a$; an angle t with the x-axis.

Exercise 15.2

1. (*i*) 3600 ft. (*ii*) 30 sec. ; 14, 400 $\sqrt{3}$ ft. (*iii*) 544 $\sqrt{3}$ ft./sec., $\tan^{-1}\frac{8}{15}$. 2. 80 ft./sec.
3. 72 ft. ; 15° or 75°. 4. $9600\sqrt{3}$ ft. 5. 3 ft. $5\frac{2}{3}$ in.
7. $60\sqrt{3}$ ft. 8. 78 ft./sec. at an angle $\tan^{-1}\frac{5}{12}$ with the horizon. 9. $\frac{\pi u^4}{g^2}$ 13. 4110 ft./sec. ; 25 miles.

Statics

Exercise 16.1

1. P lbs. wt. at right angles to the first force.
2. 120°. 3. 20 lbs. wt., 4 lbs. wt. 4. 60°.

Exercise 16.3

1. (*i*) yes (*ii*) no (*iii*) yes.
10. $2\frac{4}{5}$ lbs. wt. ; $9\frac{3}{5}$ lbs. wt.

Exercise 17.1

1. 40 kgs. wt., 10 kgs. wt.
2. $43\frac{1}{3}$ and $13\frac{1}{3}$ lbs. wt.
3. 650 lbs. 5. 12 lbs.
6. At a point dividing it in the ratio 53 : 258.
10. The pressure varies inversely as the distance between his hand and shoulder.
11. $l - a(W - P)/W$. 12. $\frac{Pd}{P+Q}$.

Exercise 18.1

1. $\frac{3}{2}AC$; $4AC$; $\frac{5}{2}AB$. 2. 13 lbs. wt. ; 17 lbs. wt.
3. $4\frac{1}{2}$ tons wt. ; $3\frac{1}{2}$ tons wt.
4. The resultant is $P\sqrt{12}$ inclined at an angle of 30° to BC and passing through Q where $CQ =$ BC.
9. $3\frac{1}{3}$ ft. ; 0 and 160 lbs. wt.
10. The required height is $\frac{1}{2}l\sqrt{2}$ where l is the length of the rope.

Exercise 19.1

1. $5a$ where a is the side of the square.
2. A couple of moment $8a$ where a is the side of the square.
5. A force equal, parallel and opposite to the force at C and acting at a point C' in AC, such that $CC' = \frac{2}{3}AB$.
11. Required force is equal and parallel to DA, at a distance from it and $= \frac{1}{4}$ the distance of B or C.

Exercise 20.1

1. $(-2, 0)$.
2. The *C.G.* divides the line joining the middle points of *AB* and *CD* in the ratio 19 : 11.
3. $8\frac{1}{3}$ cm. ; $11\frac{1}{3}$ cm. 5. 8.4 ft. ; 3 ft.

6. $\left(\frac{3}{5}b, 0\right)$. 7. $\left(\frac{2a}{3}\cdot\frac{\sin\alpha}{\alpha}, 0\right)$. 8. $\left(\frac{4a}{3\pi}, 0\right)$.

10. $\bar{x}=\bar{y}=\frac{256a}{315\pi}$. 11. $\left(\frac{\pi}{2}, \frac{\pi}{8}\right)$. 12. $\left(0, \frac{\pi}{8}\right)$.

13. $\left(\frac{a}{5}, \frac{b}{5}\right)$. 14. $\left(\frac{3a}{5}, \frac{3a}{4}\right)$.

Exercise 21.1

1. $P(x+y-1)+Q(y-x-1)-\sqrt{2}\,R(y-2)=0$.
2. $-\frac{1}{3}$. 3. $(8, 0)$, -17 and 23 ft. lbs.
5. $(3a, o)$, $(o, 3a)$.

6. $$\left\{\frac{5(P^2+Q^2+R^2)-6QR+8PR}{5}\right\},$$
$$5Px-5Qy+(4x+3y-5)=0.$$

7. A straight line.

10. $\sqrt{53+6\sqrt{3}}$ lbs. wt. ; $\frac{(3\sqrt{3}+2)a}{2}$ ft. lbs where a is the side of the lamina.

Exercise 22.1

1. 6.2 lbs. wt. at 76°6′ to *AB* cutting *AB* between *A* and *B* at a distance of $\frac{6}{7}$ inch from *A*.
4. 1.6 lbs. wt. ; 6.4 lbs. wt. ; 8 ft. from *A*.
5. 7.15 and 6.85 cwt.
6. 27.5 lbs. wt. ; 24.5 lbs. wt.
7. 34.2 and 6.8 lbs. wt.
8. 15.7 and 29.3 lbs. wt.

Exercise 23.1

1. $\frac{1}{\sqrt{3}}$; $2\sqrt{3}$ lbs. wt. 2. 4 lbs. wt. 3. $\frac{4}{3}\sqrt{3}$ lbs. wt.
4. 20 lbs. wt.
5. 10 lbs. wt. ; $12\frac{1}{2}$ lbs. wt. 9. 45°.

Exercise 24.1

1. $74\frac{2}{3}$ lbs. acting downwards. 2. 4 ft. from the first weight ; towards the first weight. 3. 8 lb., 10 lb. 4. $P=9$ lb ; $Q=15$ lb. 8. 3 lb ; 3 ft. 9. $9\frac{4}{5}$ lbs. 12. (*i*) 1, (*ii*) 0. 13. $P : Q = b\sin\beta : a\sin\alpha$; $\{P^2+Q^2-2PQ\cos(\alpha+\beta)\}^{\frac{1}{2}}$.

Exercise 24.2

1. (*i*) $45\frac{1}{3}$ kg. ; 40 kg. 2. (*i*) $10\sqrt{3}$ kg. ; 20 kg. (*ii*) $5\sqrt{3}$ kg., 5 kg.

3. 30°, $W\frac{\sqrt{3}}{2}$. 4. 5 lb. ; $\frac{5}{2}\sqrt{3}$ lb.

7. $5 : 3\cdot 4$; $2W$. 8. $\frac{1}{\sqrt{3}}$ lbs. wt. ; $\frac{7}{\sqrt{3}}$ lbs. wt.

9. $\sqrt{3} : 1$; $T=\frac{W}{2}=W'\frac{\sqrt{3}}{2}$. 13. 30° ; 45° ; 60°.

15. 30° to the horizon ; 45° above the plane. 16. $\frac{3}{4}$.

Exercise 24.3

1. $3\frac{57}{245}$ in. ; $\frac{18}{35}$ in. 2. $4\frac{94}{99}$ lbs. wt.
3. $50\frac{10}{11}$ lbs. wt. 4. $6\frac{3}{7}$ tons. wt. 5. $\frac{33}{49}$ inch.
6. $4\frac{92}{175}$ ins. 7. 224. 8. $5430\frac{6}{7}$,

Exercise 24.4

1. 320. lbs. 2. $\frac{1}{16}$; 20 lbs. wt.
3. 12.875 kgs. wt. 4. 14 lbs. wt.
5. 6 cwt. 6. 4 strings ; 2 lbs.
7. 9 stone wt. 8. 3 movable pulleys.
9. 8.9 times his own weight. 10. $4w$; $21w$.
11. 22 lbs. 12. 64 stone. 13. 7 lbs.
14. First system.
15. $\frac{11}{15}$ in. from the end.
16. 10 lbs. wt. ; the point required divides the distance between the first two strings (passing over the two topmost pulleys) in the ratio 5 : 2.
17. 8 lbs. ; 4 lbs. ; 15 lbs.
23. $W=P(2^n-1)+W'(2^{n-1}-1)$.
26. $\frac{W}{n+1}$ where n is the number of strings ; $\frac{W}{n-1}$.
28. $\frac{1}{2}Wl/\sqrt{l^2-d^2\cos^2\alpha}$.
33. $W=P(2^{2n+1}+2^n)-w(2^{2n+1}-3.2^n+n+2)$.

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